18.02 Practice final-Solutions

Problem 1.

$P:(1,1,-1), \quad Q:(1,2,0), \quad R:(-2,2,2)$

$\overrightarrow{PQ} = <0,1,1>, \overrightarrow{PR} = <-3,1,3> \qquad \overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ -3 & 1 & 3 \end{vmatrix}$

Plane: $\boxed{2x-3y+3z=-4}$ (substitute any of the pts. into 2x-3y+3z=d)

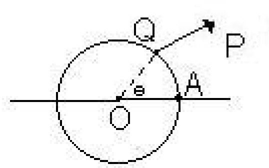
Problem 2.

$$\begin{vmatrix} 1 & 0 & c \\ 2 & c & 1 \\ 1 & -1 & 2 \end{vmatrix} = (2c-2c)-(c^2-1) = 1-c^2 \quad \therefore \quad |\quad| = 0 \Leftrightarrow \boxed{c=\pm 1}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & \boxed{1} \\ 1 & -1 & 2 \end{bmatrix} \text{ cofactor } \boxed{1} = -\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = 1, \text{ det} = 1-2^2=-3 \quad \therefore \quad \boxed{-\frac{1}{3}}$$

Problem 3.

$\overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QP}, \quad \overrightarrow{OQ} = a<\cos\theta, \sin\theta>, \quad \overrightarrow{QP} = a\theta\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle$



$$\therefore \quad \boxed{x = a(\cos\theta + \frac{\theta\sqrt{2}}{2}), \quad y = a(\sin\theta + \frac{\theta\sqrt{2}}{2})}$$

Problem 4.

$\vec{r} = <3\cos t, 5\sin t, 4\cos t> \qquad \vec{v} = <-3\sin t, 5\cos t, -4\sin t>$

$|\vec{v}| = \sqrt{9\sin^2 t + 16\sin^2 t + 25\cos^2 t} = \boxed{5}$. Passes through yz plane when $x=0$,

$\therefore$ when $\cos t = 0: \quad t = \frac{\pi}{2}, \frac{3\pi}{2} \qquad \therefore$ at $(0,\pm 5, 0)$

Problem 5.

$\omega = x^2y - xy^3, P=(2,1)$

a) $\overrightarrow{\nabla\omega} = (2xy-y^3)i + (x^2-3xy^2)j$

$\overrightarrow{\nabla\omega}_p = 3i-2j, \left(\frac{d\omega}{ds}\right)_p = (3i-2j)\cdot\frac{3i+4j}{5} = \boxed{\frac{1}{5}}$

b) $\frac{\Delta\omega}{\Delta s} \approx \frac{1}{5}, \quad \therefore \quad \Delta\omega \approx \frac{1}{5}(.01) = \boxed{.002}$

Problem 6. $x^2 + 2y^2 + 2z^2 = 5$, $\overrightarrow{\nabla\omega} =< 2x, 4y, 4z >=< 2, 4, 4 >$ at $(1, 1, 1)$

tan. plane: $\boxed{x + 2y + 2z = 5}$, dihedral angle θ (angle between normals) :

$\cos\theta = \dfrac{< 1, 2, 2 >}{3} \cdot \hat{k} = \dfrac{2}{3} \qquad \therefore \qquad \boxed{\theta = \cos^{-1}(2/3)}$

Problem 7.

Minimize $x^2 + y^2 + z^2$, with $2x + y - z - 6 = 0 \quad \oplus$

$$\text{Lagrange equations:} \quad \begin{array}{rcl} 2x & = & 2\lambda \\ 2y & = & \lambda \\ 2z & = & -\lambda \end{array} \quad \text{substituting into } \oplus\text{: } 2\lambda + \tfrac{\lambda}{2} - \left(\tfrac{-\lambda}{2}\right) = 6$$

$\therefore \qquad \lambda = 2.$ Ans: $(2, 1, -1)$

Problem 8.

$g(x, y, z) = 3, \quad (\overrightarrow{\nabla g})_p =< 2, -1, -1 > \quad \therefore \quad g_x + g_z \cdot \frac{\partial z}{\partial x} = 0$; at P,

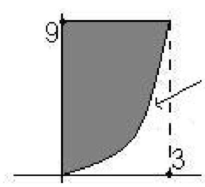
a) $\dfrac{\partial z}{\partial x} = \dfrac{-g_x}{g_z} = \dfrac{-2}{-1} = \boxed{2}$

b) $\left(\dfrac{\partial \omega}{\partial x}\right)_y = (f_x)\left(\dfrac{\partial x}{\partial x}\right)_y + (f_y)\left(\dfrac{\partial y}{\partial x}\right)_y + (f_z)\left(\dfrac{\partial z}{\partial x}\right)_y = 1 \cdot 1 + 1 \cdot 0 + 2 \cdot 2 = \boxed{5}$

Problem 9.

$$\int_0^3 \int_{x^2}^9 x e^{-y^2} dy dx = \int_0^9 \int_0^{\sqrt{y}} x e^{-y^2} dx dy$$

Inner : $\left[\dfrac{1}{2} x^2 e^{-y^2}\right]_0^{\sqrt{y}} = \dfrac{1}{2} y e^{-y^2}$,



$$\begin{array}{rcl} y & = & x^2 \\ x & = & \sqrt{y} \end{array}$$

Outer : $\left[-\dfrac{e^{-y^2}}{4}\right]_0^9 = \dfrac{1}{4}\left[1 - e^{-81}\right]$

Problem 10.

Circle is $r = 2\cos\theta$. Integrate over $\dfrac{1}{8}$ region: $8 \displaystyle\int_0^{\frac{\pi}{4}} \int_0^{2\cos\theta} r^2 \cdot r dr d\theta$

$\left[\text{or} \quad 4 \displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int \ldots\right]$

Problem 11.

$\displaystyle\oint P dy - Q dx \quad \left[\text{or} \quad \oint -Q dx + P dy\right]$

b) By Green's Thm: above$= \displaystyle\int\int_R (P_x + Q_y) dx dy = \int\int_R (a + b) dx dy =$ area of R

$\Leftrightarrow \boxed{a + b = 1}$

Problem 12.

$F = G \int\int\int \frac{\cos\phi}{\rho^2} \cdot \delta \cdot \rho^2 \sin\phi \, d\rho d\phi d\theta$

$\delta = z = \rho\cos\phi \quad \therefore \quad F = G\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 \rho\cos^2\phi\sin\phi \, d\rho d\phi d\theta =$

$= G\cdot 2\pi \cdot \int_0^{\frac{\pi}{2}} \cos^2\phi\sin\phi d\phi \cdot \int_0^1 \rho d\rho = G\cdot 2\pi\cdot\left[\frac{-\cos^3\phi}{3}\right]_0^{\frac{\pi}{2}} \cdot \left[\frac{1}{2}\rho^2\right]_0^1 =$

$= 2\pi G\cdot\frac{1}{3}\cdot\frac{1}{2} = \boxed{\frac{\pi G}{3}}$

Problem 13.

Line from $P:(1,1,1)$ to $Q:(2,4,8)$ is:

$x = 1+t, \quad y = 1+3t, \quad z = 1+7t$ (since $\overrightarrow{PQ} =< 1,3,7 >$) $0 \le t \le 1. \quad \therefore$

$\int_C (y-x)dx + (y-z)dz = \int_0^1 2tdt - 4\cdot 7tdt = \int_0^1 -26tdt = \left[-13t^2\right]_0^1 = \boxed{-13}$

Problem 14.

a) $\vec{F} =< ay^2, 2yx+2yz, by^2+z^2 >$

Test: $2ay = 2y \quad \therefore \quad a = 1,\ 2y = 2by \quad \therefore \quad b = 1,\ 0 = 0$

b) By any method, $f(x,y,z) = \boxed{xy^2 + y^2z + \frac{z^3}{3}}$

c) Any surface S: $\boxed{xy^2 + y^2z + \frac{z^3}{3} = C}$

Problem 15.

$\oint\!\!\oint_S \vec{F}\cdot d\vec{S} = \int\!\!\int\!\!\int_V div\vec{F}\, dV. \quad \therefore \quad \int\!\!\int_B \vec{F}\cdot d\vec{S} + \int\!\!\int_U \vec{F}\cdot d\vec{S} = \int\!\!\int\!\!\int_V dV = 3V.$

Volume $V = \int_0^{2\pi}\int_0^1 (1-r^2)rdrd\theta = 2\pi\left[\frac{r^2}{2} - \frac{1}{4}\right]_0^1 = \frac{\pi}{2},$

$\int\!\!\int_B = 0$ since $\vec{F}\cdot d\vec{S} = z = 0$ on xy-plane $\therefore \quad \int\!\!\int_U \vec{F}\cdot d\vec{S} = \boxed{\frac{3\pi}{2}}$

Problem 16.

$\vec{F} =< x,y,z >, \quad z = 1 - x^2 - y^2$

$\hat{n}dS =< -f_x, -f_y, 1 > dxdy =< 2x, 2y, 1 > dx$

$\vec{F}\cdot\hat{n}dS = (2x^2+2y^2+z)dxdy = (x^2+y^2+1)dxdy$ $\therefore$ flux over U is:

$\int\!\!\int (x^2+y^2+1)dxdy = \int_0^{2\pi}\int_0^1 (r^2+1)rdrd\theta = 2\pi\left[\frac{r^4}{4} + \frac{r^2}{2}\right]_0^1 = 2\pi\cdot\frac{3}{4} = \boxed{\frac{3\pi}{2}}$

Problem 17.

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \vec{\nabla}\times\vec{F}\cdot d\vec{S}, \qquad \vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^2 & y^2 & xz \end{vmatrix} = -zj$$

The normal vector to $f(x,z)=0$ is $\hat{n} = \dfrac{\overrightarrow{\nabla f}}{|\overrightarrow{\nabla f}|} = \dfrac{f_x i + f_z \hat{k}}{|\overrightarrow{\nabla f}|}$

$\therefore\ \vec{\nabla}\times\vec{F}\cdot\hat{n} = 0$, so $\displaystyle\oint_C \vec{F}\cdot d\vec{r} = 0$

Problem 18.

$$\int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2}\, dydx$$

a) $\displaystyle = \int_0^\infty e^{-x^2}dx \int_0^\infty e^{-y^2}dy = I\cdot I$

b) $\displaystyle = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\cdot r dr d\theta = \pi/2\cdot\left[\frac{e^{-r^2}}{-2}\right]_0^\infty = \frac{\pi}{2}\cdot\frac{1}{2} = \frac{\pi}{4}$

$I^2 = \dfrac{\pi}{4} \quad \therefore \quad I = \boxed{\dfrac{\sqrt{\pi}}{2}}$