Integrals in Spherical Coordinates

1. Find the volume of a sphere of radius a.

<u>Answer:</u> From the problems on limits in spherical coordinates (Session 76), we have limits: inner ρ: 0 to a –radial segments

middle ϕ: 0 to π –fan of rays.

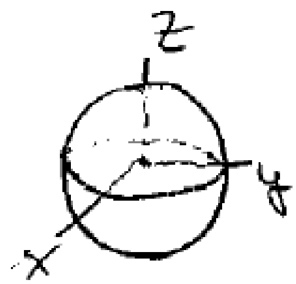
outer θ: 0 to 2π –volume.

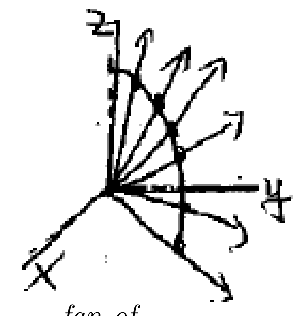
$$V = \iiint_D dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Inner: $\left. \dfrac{\rho^3}{3} \sin\phi \right|_0^a = \dfrac{a^3}{3} \sin\phi$

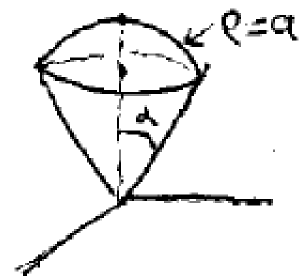
Middle: $\left. -\dfrac{a^3}{3} \cos\phi \right|_0^{\pi} = \dfrac{2}{3} a^3$

Outer: $\frac{4}{3}\pi a^3$ –as it should be.





2. Find the centroid of the region bounded by the sphere $\rho = a$ and the cone $\phi = \alpha$.



<u>Answer:</u> In Session 76 we computed the limits:

ρ: 0 to a, ϕ: 0 to α, θ: 0 to 2π.

By symmetry, $x_{cm} = y_{cm} = 0$.

$$z_{cm} = \frac{1}{V} \iiint_D z \, dV = \frac{1}{V} \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho\cos\phi \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{1}{V} \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^3 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta.$$

Inner, middle and outer integrals are easy to compute: $z_{cm} = \dfrac{1}{V} \cdot \dfrac{\pi a^4 \sin^2\alpha}{4}$.

$$V = \iiint_D dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{2}{3}\pi a^3 (1 - \cos\alpha).$$

$$\Rightarrow \; z_{cm} = \frac{a^4 \sin^2\alpha \, \pi}{4} \cdot \frac{3}{2\pi a^3 (1 - \cos\alpha)} = \frac{3a}{8} \cdot \frac{\sin^2\alpha}{1 - \cos\alpha} = \frac{3}{8} a (1 + \cos\alpha).$$