Triple Integrals

1. Find the moment of inertia of the tetrahedron shown about the z-axis. Assume the tetrahedron has density 1.

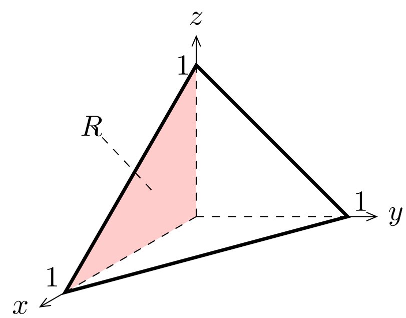

Figure 1: The tetrahedron bounded by $x + y + z = 1$ and the coordinate planes.

Answer: To compute the moment of inertia, we integrate distance squared from the z-axis times mass:

$$\iiint_D (x^2 + y^2) \cdot 1 \, dV.$$

Using cylindrical coordinates about the axis of rotation would give us an "easy" integrand (r) with complicated limits. The integrand $x^2 + y^2$ is not particularly intimidating, so we instead use rectangular coordinates. Integrating first with respect to y or x is preferable; $(x^2 + y^2)(1 - x - y)$ is a somewhat more intimidating integrand.

To find our limits of integration, we let y go from 0 to the slanted plane $x + y + z = 1$. The x and z coordinates are in R, the *projection* of D to the xz-plane which is bounded by the x and z axes and the line $x + z = 1$.

$$\text{Moment of Inertia } = \int_0^1 \int_0^{1-z} \int_0^{1-x-z} (x^2 + y^2) \, dy \, dx \, dz.$$

Inner: $(x^2 y + \frac{1}{3} y^3) \Big|_0^{1-x-z} = x^2 - x^3 - x^2 z + \frac{1}{3}(1 - x - z)^3.$

Middle:

$$\begin{aligned} \int_0^{1-z} x^2(1-z) - x^3 + \frac{1}{3}(1-x-z)^3 \, dx &= \frac{1}{3}x^3(1-z) - \frac{1}{4}x^4 - \frac{1}{12}(1-x-z)^4 \Big|_0^{1-z} \\ &= \frac{1}{3}(1-z)^4 - \frac{1}{4}(1-z)^4 + \frac{1}{12}(1-z)^4 \\ &= \frac{1}{6}(1-z)^4. \end{aligned}$$

Outer: $\frac{1}{30}(1-z)^5 \Big|_0^1 = \frac{1}{30}.$

2. Find the mass of a cylinder centered on the z-axis which has height h, radius a and density $\delta = x^2 + y^2$.

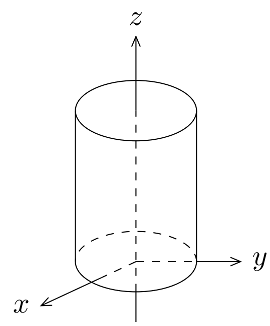

Figure 2: Cylinder.

Answer: To find the mass we integrate the product of density and volume:

$$\text{Mass} = \iiint_D \delta \, dV = \iiint_D r^2 \, dV.$$

Naturally, we'll use cylindrical coordinates in this problem. The limits on z run from 0 to h. The x and y coordinates lie in a disk of radius a, so $0 \le r \le a$ and $0 < \theta \le 2\pi$.

$$\text{Mass} = \iiint_D r^2 \, dV = \int_0^{2\pi} \int_0^a \int_0^h r^2 \, dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^a \int_0^h r^3 \, dz \, dr \, d\theta.$$

Inner integral: $r^3 z\big|_0^h = hr^3$.

Middle integral: $\displaystyle\int_0^a hr^3 \, dr = \frac{ha^4}{4}$.

Outer integral: $2\pi \dfrac{ha^4}{4} = \dfrac{\pi h a^4}{2}$.

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