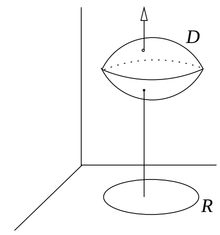


Limits in Iterated Integrals

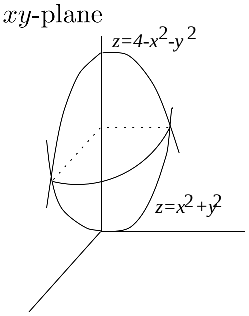
3. Triple integrals in rectangular and cylindrical coordinates.

You do these the same way, basically. To supply limits for $\iiint_D dz\,dy\,dx$ over the region D, we integrate first with respect to z. Therefore we

1. Hold x and y fixed, and let z increase. This gives us a vertical line.

2. Integrate from the z-value where the vertical line enters the region D to the z-value where it leaves D.

3. Supply the remaining limits (in either xy-coordinates or polar coordinates) so that you include all vertical lines which intersect D. This means that you will be integrating the remaining double integral over the region R in the xy-plane which D projects onto.



For example, if D is the region lying between the two paraboloids

$$z = x^2 + y^2 \qquad\qquad z = 4 - x^2 - y^2,$$

we get by following steps 1 and 2,

$$\iiint_D dz\,dy\,dx \;=\; \iint_R \int_{x^2+y^2}^{4-x^2-y^2} dz\,dA$$

where R is the projection of D onto the xy-plane. To finish the job, we have to determine what this projection is. From the picture, what we should determine is the xy-curve over which the two surfaces intersect. We find this curve by eliminating z from the two equations, getting

$$\begin{aligned} x^2 + y^2 &= 4 - x^2 - y^2, \qquad \text{which implies} \\ x^2 + y^2 &= 2. \end{aligned}$$

Thus the xy-curve bounding R is the circle in the xy-plane with center at the origin and radius $\sqrt{2}$.

This makes it natural to finish the integral in polar coordinates. We get

$$\iiint_D dz\,dy\,dx \;=\; \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{x^2+y^2}^{4-x^2-y^2} dz\,r\,dr\,d\theta \;;$$

the limits on z will be replaced by r^2 and $4 - r^2$ when the integration is carried out.

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