# 18.02 Practice Exam 4 A – Solutions

**Problem 1.**
a)

$$M_y = e^x z = N_x$$
$$M_z = e^x y = P_x$$
$$N_Z = e^x + 2y = P_y$$

b) We begin with

$$f_x = e^x yz$$
$$f_y = e^x z + 2yz$$
$$f_z = e^x y + y^2 + 1$$

Integrating $f_x$ we get $f = e^x yz + g(y, z)$. Differentiating and comparing with the above equations we get

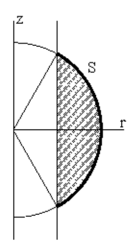
$$\begin{cases} f_y = e^x z + g_y \\ f_z = e^x y + g_z \end{cases} \rightarrow \begin{cases} g_y = 2yz \\ g_z = y^2 + 1 \end{cases}$$

Integrating $g_y$ we get $g = y^2 z + h(z)$. Then $g_z = y^2 + h'(z)$ so comparing with the second equation we get $h'(z) = 1$. Hence $h = z + C$. Putting everything together we get

$$f = e^x yz + y^2 z + z + C.$$

1c) $N_z = 0$ and $P_y = 1$ hence the field is not conservative.

**Problem 2.**
a) Consider the figure




$\mathbf{n} = \frac{1}{2}\langle x, y, z\rangle$ hence

$$\mathbf{F} \cdot \mathbf{n} = \langle y, -x, z\rangle \cdot \frac{\langle x, y, z\rangle}{2} = \frac{z^2}{2}$$

$z = 2\cos\phi$ and $dS = 2^2 \sin\phi \, d\phi \, d\theta$ hence we get

$$\int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{4\cos^2\phi}{2} 4\sin\phi \, d\phi \, d\theta = 16\pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \cos^2\phi \sin\phi \, d\phi = 16\pi \left[\frac{\cos^3\phi}{3}\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} = 4\sqrt{3}\,\pi$$

b) $\mathbf{n} = \pm\langle x, y, 0\rangle$ hence $\mathbf{F} \cdot \mathbf{n} = 0$. So the flux is 0.
c) $div\mathbf{F} = 1$ hence

$$Vol(R) = \int\int\int_R 1 \, dV = \int\int\int div\mathbf{F} \, dV = \int\int_S \mathbf{F} \cdot \mathbf{n} \, dS + \int\int_{\text{Cylinder}} \mathbf{F} \cdot \mathbf{n} \, dS = 4\sqrt{3}\,\pi$$

**Problem 3.**

a) $C$ is given by the equations $x^2 + y^2 + z^2 = 2$ and $z = 1$. So $x^2 + y^2 = 1$.
Parametrization:

$$x = cost \qquad y = \sin t \qquad z = 1$$
$$dx = -\sin t\, dt \qquad dy = \cos t\, dt \qquad dz = 0$$

So

$$I = \int_0^{2\pi} (-\cos t \sin t + \sin t \cos t)\, dt = 0.$$

b)

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ xz & y & y \end{vmatrix} = \hat{\mathbf{i}} + x\hat{\mathbf{j}}$$

c) By Stokes' theorem

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$$

$\mathbf{n}$ is the normal pointing upward hence

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int\int_S \langle 1, x, 0\rangle \cdot \frac{\langle x, y, z\rangle}{\sqrt{2}}\, dS = \int\int_S \frac{x + xy}{\sqrt{2}}\, dS$$

**Problem 4.** $div\mathbf{F} = 0$ hence

$$\int\int_S \mathbf{F} \cdot \mathbf{n}\, dS = \int\int\int div\mathbf{F}\, dV = 0$$

**Problem 5.**

a) $z = (x^2 + y^2 + z^2)^2 \geq 0$

b) $z = \rho \cos \phi$ and $x^2 + y^2 + z^2 = \rho^2$ hence $\rho \cos \phi = \rho^4$. Canceling $\rho$ we get $\cos \phi = \rho^3$.

c)

$$\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{(\cos\phi)^{1/3}} \rho^2 \sin \phi\, d\rho\, d\phi\, d\theta$$

**Problem 6.** The flux is upward so and $z = f(x, y) = xy$, hence

$$\mathbf{n}\, dS = +\langle -f_x, -f_y, 1\rangle\, dx\, dy = \langle -y, -x, 1\rangle\, dx\, dy$$

So,

$$\int\int_S \mathbf{F} \cdot \mathbf{n}\, dS = \int\int_{x^2+y^2<1} \langle y, x, z\rangle \cdot \langle -y, -x, 1\rangle = \int\int_{x^2+y^2<1} (-y^2 - x^2 + xy)\, dx\, dy$$

where we substituted $z = xy$. Using polar coordinates we get

$$\int_0^{2\pi} \int_0^1 (-r^2 + r^2 \cos\theta \sin\theta)\, r\, dr\, d\theta$$

Inner: $\int_0^1 (-r^2 + r^2 \cos\theta \sin\theta)\, r\, dr = \frac{1}{4}(\cos\theta \sin\theta - 1)$

Outer: $\int_0^{2\pi} \frac{1}{4}(\cos\theta \sin\theta - 1)\, d\theta = \frac{1}{4}\left[\frac{\sin^2\theta}{2} - \theta\right]_0^{2\pi} = -\frac{\pi}{2}$