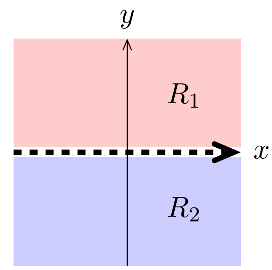
Problems: Extended Green's Theorem

1. Is $\mathbf{F} = \dfrac{y\,dx - x\,dy}{y^2}$ exact? If so, find a potential function.



<u>Answer:</u> $M = \dfrac{1}{y}$ and $N = -\dfrac{x}{y^2}$ are continuously differentiable whenever $y \neq 0$, i.e. in the two half-planes R_1 and R_2 – both simply connected.
Since $M_y = -1/y^2 = N_x$ in each half-plane the field is exact where it is defined.
To find a potential function f for which $\mathbf{F} = df$ we use method 2.
$f_x = 1/y \;\Rightarrow\; f = x/y + g(y)$.
$f_y = -x/y^2 + g'(y) = -x/y^2 \;\Rightarrow\; g'(y) = 0 \;\Rightarrow\; g(y) = c$.
$\Rightarrow\; f(x, y) = x/y + c$.

(continued)

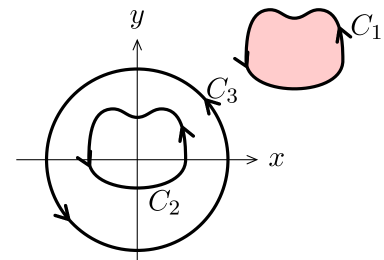
Example 3: Let $\mathbf{F} = r^n(x\mathbf{i} + y\mathbf{j})$. Use extended Green's Theorem to show that $\mathbf{F}$ is conservative for all integers n. Find a potential function.

First note, $M = r^n x$, $N = r^n y \;\Rightarrow\; M_y = nr^{n-2}xy = N_x \;\Leftrightarrow\;$ curl$\mathbf{F} = 0$.

We show $\mathbf{F}$ is conservative by showing $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for all simple closed curves C.

If C_1 is a simple closed curve not around 0 then Green's Theorem implies $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = 0$.

If C_3 is a circle centered on (0,0) then, since $\mathbf{F}$ is radial $\oint_{C_3} \mathbf{F}\cdot d\mathbf{r} = \oint_{C_3} \mathbf{F}\cdot\mathbf{T}\,ds = 0$.



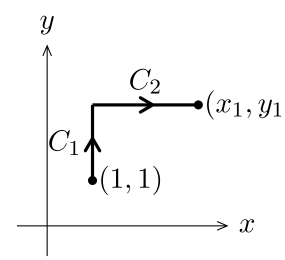
If C_3 completely surrounds C_2 then extended Green's Theorem implies $\oint_{C_2} \mathbf{F}\cdot d\mathbf{r} = \oint_{C_3} \mathbf{F}\cdot d\mathbf{r} = 0$.

Thus $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for all closed loops $\Rightarrow$ $\mathbf{F}$ is conservative.

To find the potential function we use method 1 over the curve C shown.

The calculation works for $n = 2$. For $n = 2$ everything is the same except we'd get natural logs instead of powers. (We also ignore the fact that if (x_1, y_1) is on the negative x-axis we shoud use a different path that doesn't go through the origin. This isn't really an issue since we already know a potential function exists, so continuity would handle these points without using an integral.)

$$\begin{aligned}
f(x_1,y_1) &= \int_C r^n x\,dx + r^n y\,dy \\
&= \int_1^{y_1} (1+y^2)^{n/2} y\,dy + \int_1^{x_1} (x^2+y_1^2)^{n/2} x\,dx \\
&= \left.\frac{(1+y^2)^{(n+2)/2}}{n+2}\right|_1^{y_1} + \left.\frac{(x^2+y_1^2)^{(n+2)/2}}{n+2}\right|_1^{x_1} \\
&= \frac{(1+y_1^2)^{(n+2)/2} - 2^{(n+2)/2}}{n+2} + \frac{(x_1^2+y_1^2)^{(n+2)/2} - (1+y_1^2)}{(n+2)/2} \\
&= \frac{(x_1^2+y_1^2)^{(n+2)/2} - 2^{(n+1)/2}}{n+2}
\end{aligned}$$



$$\Rightarrow \boxed{f(x,y) = \frac{r^{n+2}}{n+2} + C.}$$

If $n = -2$ we get $f(x,y) = \ln r + C$.

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