Problems: Green's Theorem and Area

1. Find M and N such that $\oint_C M\,dx + N\,dy$ equals the polar moment of inertia of a uniform density region in the plane with boundary C.

Answer: Let R be the region enclosed by C and ρ be the density of R. The polar moment of inertia is calculated by integrating the product mass times distance to the origin:

$$I = \iint_R dI = \iint_R r^2\,dm = \iint_R (x^2 + y^2) \cdot \rho\,dA.$$

Green's theorem now tells us that we're looking for functions M and N such that $N_x - M_y = \rho x^2 + \rho y^2$. The simplest choice is $N_x = \rho y^2$, $M_y = -\rho x^2$. This leads to $N = \rho x y^2$, $M = -\rho x^2 y$.

Use Green's theorem to check this answer:

$$\begin{aligned}\oint_C -\rho x^2 y\,dx + \rho x y^2\,dy &= \iint_R \rho y^2 - (-\rho x^2)\,dA \\ &= \iint_R r^2 \cdot \rho\,dA = I.\end{aligned}$$