

## Using Green's Theorem

1. Show that  $\oint_C -x^2y dx + xy^2 dy > 0$  for all simple closed curves  $C$ .

**Answer:**

If  $R$  is the interior of  $C$ , then Green's Theorem tells us:

$$\oint_C M dx + N dy = \iint_R N_x - M_y dA.$$

Here,  $M = -x^2y$  and  $N = xy^2$ , so  $N_x - M_y = y^2 - (-x^2) = x^2 + y^2$ . In other words,  $N_x - M_y$  is the square of the distance from  $(x, y)$  to the origin. This distance is always positive, so the integral of this value over any non-empty region in the plane will be positive.

We conclude that  $\oint_C -x^2y dx + xy^2 dy = \iint_R (x^2 + y^2) dA > 0$ .

2. Let  $\mathbf{F} = 2y\mathbf{i} + x\mathbf{j}$  and let  $C$  be the positively oriented unit circle. Compute  $\oint_C \mathbf{F} \cdot d\mathbf{r}$  directly and by Green's theorem.

**Answer:** Using Green's theorem:  $N_x - M_y = 1 - 2 = -1$ , so  $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (-1) dA = -\pi$ .

Directly (using half-angle formulas): We parametrize  $C$  by  $x = \cos t$ ,  $y = \sin t$ ,  $0 \leq t \leq 2\pi$ . Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} -2\sin^2 t + \cos^2 t dt = \int_0^{2\pi} -(1 - \cos 2t) + \frac{1 + \cos 2t}{2} dt = -\pi.$$

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