Using Green's Theorem

1. Show that $\oint_C -x^2y\,dx + xy^2\,dy > 0$ for all simple closed curves C.

Answer:

If R is the interior of C, then Green's Theorem tells us:

$$\oint_C M\,dx + N\,dy = \iint_R N_x - M_y\,dA.$$

Here, $M = -x^2y$ and $N = xy^2$, so $N_x - M_y = y^2 - (-x^2) = x^2 + y^2$. In other words, $N_x - M_y$ is the square of the distance from (x, y) to the origin. This distance is always positive, so the integral of this value over any non-empty region in the plane will be positive.

We conclude that $\oint_C -x^2y\,dx + xy^2\,dy = \iint_R (x^2 + y^2)\,dA > 0$.

2. Let $\mathbf{F} = 2y\mathbf{i} + x\mathbf{j}$ and let C be the positively oriented unit circle. Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly and by Green's theorem.

Answer: Using Green's theorem: $N_x - M_y = 1 - 2 = -1$, so $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (-1)\,dA = -\pi$.

Directly (using half-angle formulas): We parametrize C by $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} -2\sin^2 t + \cos^2 t\,dt = \int_0^{2\pi} -(1 - \cos 2t) + \frac{1 + \cos 2t}{2}\,dt = -\pi.$$