Changing Variables in Multiple Integrals

3. Examples and comments; putting in limits.

If we write the change of variable formula as

$$\iint_R f(x,y)\,dx\,dy \;=\; \iint_R g(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv\ , \tag{18}$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} \;=\; \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}\ , \qquad g(u,v) = f(x(u,v), y(u,v)), \tag{19}$$

it looks as if the essential equations we need are the inverse equations:

$$x = x(u,v), \qquad y = y(u,v) \tag{20}$$

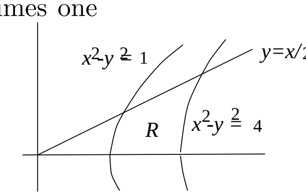
rather than the direct equations we are usually given:

$$u = u(x,y), \qquad v = v(x,y)\ . \tag{21}$$

If it is awkward to get (20) by solving (21) simultaneously for x and y in terms of u and v, sometimes one can avoid having to do this by using the following relation (whose proof is an application of the chain rule, and left for the Exercises):

$$\frac{\partial(x,y)}{\partial(u,v)}\,\frac{\partial(u,v)}{\partial(x,y)} \;=\; 1 \tag{22}$$

The right-hand Jacobian is easy to calculate if you know $u(x,y)$ and $v(x,y)$; then the left-hand one — the one needed in (19) — will be its reciprocal. Unfortunately, it will be in terms of x and y instead of u and v, so (20) still ought to be needed, but sometimes one gets lucky. The next example illustrates.



Example 3. Evaluate $\displaystyle\iint_R \frac{y}{x}\,dx\,dy$, where R is the region pictured, having as boundaries the curves $x^2 - y^2 = 1$, $\quad x^2 - y^2 = 4$, $\quad y = 0$, $\quad y = x/2$.

Solution. Since the boundaries of the region are contour curves of $x^2 - y^2$ and y/x , and the integrand is y/x, this suggests making the change of variable

$$u \;=\; x^2 - y^2, \qquad v \;=\; \frac{y}{x}\ . \tag{23}$$

We will try to get through without solving these backwards for x, y in terms of u, v. Since changing the integrand to the u, v variables will give no trouble, the question is whether we can get the Jacobian in terms of u and v easily. It all works out, using (22):

$$\frac{\partial(u,v)}{\partial(x,y)} \;=\; \begin{vmatrix} 2x & -2y \\ -y/x^2 & 1/x \end{vmatrix} \;=\; 2 - 2y^2/x^2 \;=\; 2 - 2v^2; \qquad \text{so} \quad \frac{\partial(x,y)}{\partial(u,v)} \;=\; \frac{1}{2(1-v^2)}\ ,$$

according to (22). We use now (18), put in the limits, and evaluate; note that the answer is positive, as it should be, since the integrand is positive.

$$\begin{aligned}\iint_R \frac{y}{x}\,dx\,dy &= \iint_R \frac{v}{2(1-v^2)}\,du\,dv \\ &= \int_0^{1/2}\int_1^4 \frac{v}{2(1-v^2)}\,du\,dv \\ &= -\frac{3}{4}\ln(1-v^2)\Big]_0^{1/2} = -\frac{3}{4}\ \ln\ \frac{3}{4}\ .\end{aligned}$$

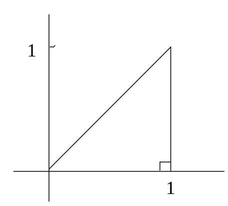
Putting in the limits

In the examples worked out so far, we had no trouble finding the limits of integration, since the region R was bounded by contour curves of u and v, which meant that the limits were constants.

If the region is not bounded by contour curves, maybe you should use a different change of variables, but if this isn't possible, you'll have to figure out the uv-equations of the boundary curves. The two examples below illustrate.

Example 4. Let $u = x + y,\ v = x - y$; change $\int_0^1\int_0^x dy\,dx$ to an iterated integral $du\,dv$.

Solution. Using (19) and (22), we calculate $\dfrac{\partial(x,y)}{\partial(u.v)} = -1/2$, so the Jacobian factor in the area element will be $1/2$.



To put in the new limits, we sketch the region of integration, as shown at the right. The diagonal boundary is the contour curve $v = 0$; the horizontal and vertical boundaries are not contour curves — what are their uv-equations? There are two ways to answer this; the first is more widely applicable, but requires a separate calculation for each boundary curve.

Method 1 *Eliminate x and y from the three simultaneous equations $u = u(x,y)$, $v = v(x,y)$, and the xy-equation of the boundary curve.* For the x-axis and $x = 1$, this gives

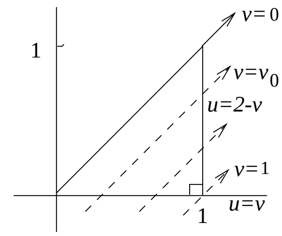
$$\begin{cases} u = x + y \\ v = x - y \\ y = 0 \end{cases} \Rightarrow \quad u = v; \qquad\qquad \begin{cases} u = x + y \\ v = x - y \\ x = 1 \end{cases} \Rightarrow \quad \begin{cases} u = 1 + y \\ v = 1 - y \end{cases} \Rightarrow \quad u + v = 2.$$

Method 2 *Solve for x and y in terms of u, v; then substitute $x = x(u,v),\ y = y(u,v)$ into the xy-equation of the curve.*

Using this method, we get $x = \frac{1}{2}(u+v),\ \ y = \frac{1}{2}(u-v)$; substituting into the xy-equations:

$$y = 0 \ \Rightarrow\ \frac{1}{2}(u-v) = 0 \ \Rightarrow\ u = v; \qquad\qquad x = 1 \ \Rightarrow\ \frac{1}{2}(u+v) = 1 \ \Rightarrow\ u + v = 2.$$

To supply the limits for the integration order $\iint du\,dv$, we



1. first hold v fixed, let u increase; this gives us the dashed lines shown;
2. integrate with respect to u from the u-value where a dashed line enters R (namely, $u = v$), to the u-value where it leaves (namely, $u = 2 - v$).
3. integrate with respect to v from the lowest v-values for which the dashed lines intersect the region R (namely, $v = 0$), to the highest such v-value (namely, $v = 1$).

Therefore the integral is $\displaystyle\int_0^1 \int_v^{2-v} \frac{1}{2}\, du\, dv$.

(As a check, evaluate it, and confirm that its value is the area of R. Then try setting up the iterated integral in the order $dv\, du$; you'll have to break it into two parts.)

Example 5. Using the change of coordinates $u = x^2 - y^2,\ v = y/x$ of Example 3, supply limits and integrand for $\displaystyle\iint_R \frac{dxdy}{x^2}$, where R is the infinite region in the first quadrant under $y = 1/x$ and to the right of $x^2 - y^2 = 1$.

Solution. We have to change the integrand, supply the Jacobian factor, and put in the right limits.

To change the integrand, we want to express x^2 in terms of u and v; this suggests eliminating y from the u, v equations; we get

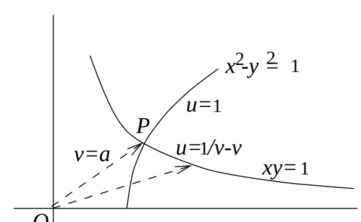
$$u = x^2 - y^2, \quad y = vx \quad \Rightarrow \quad u = x^2 - v^2x^2 \quad \Rightarrow \quad x^2 = \frac{u}{1 - v^2} \ .$$

From Example 3, we know that the Jacobian factor is $\dfrac{1}{2(1 - v^2)}$; since in the region R we have by inspection $0 \le v < 1$, the Jacobian factor is always positive and we don't need the absolute value sign. So by (18) our integral becomes

$$\iint_R \frac{dx\, dxy}{x^2} \;=\; \iint_R \frac{1 - v^2}{2u(1 - v^2)}\, du\, dv \;=\; \iint_R \frac{du\, dv}{2u}$$

Finally, we have to put in the limits. The x-axis and the left-hand boundary curve $x^2 - y^2 = 1$ are respectively the contour curves $v = 0$ and $u = 1$; our problem is the upper boundary curve $xy = 1$. To change this to $u - v$ coordinates, we follow Method 1:

$$\begin{cases} u = x^2 - y^2 \\ y = vx \\ xy = 1 \end{cases} \quad \Rightarrow \begin{cases} u = x^2 - 1/x^2 \\ v = 1/x^2 \end{cases} \quad \Rightarrow \ u = \frac{1}{v} - v \ .$$



The form of this upper limit suggests that we should integrate first with respect to u. Therefore we hold v fixed, and let u increase; this gives the dashed ray shown in the picture; we integrate from where it enters R at $u = 1$ to where it leaves, at $u = \dfrac{1}{v} - v$.

The rays we use are those intersecting R: they start from the lowest ray, corresponding to $v = 0$, and go to the ray $v = a$, where a is the slope of OP. Thus our integral is

$$\int_0^a \int_1^{1/v - v} \frac{du\, dv}{2u} \ .$$

To complete the work, we should determine a explicitly. This can be done by solving $xy = 1$ and $x^2 - y^2 = 1$ simultaneously to find the coordinates of P. A more elegant approach is to add $y = ax$ (representing the line OP) to the list of equations, and solve all three simultaneously for the slope a. We substitute $y = ax$ into the other two equations, and get

$$\begin{cases} ax^2 = 1 \\ x^2(1-a^2) = 1 \end{cases} \quad \Rightarrow \quad a = 1 - a^2 \quad \Rightarrow \quad a = \frac{-1+\sqrt{5}}{2} \ ,$$

by the quadratic formula.