Problems: Polar Coordinates and the Jacobian

1. Let $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$. Directly calculate the Jacobian $\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$.

Answer: Because we are familiar with the change of variables from rectangular to polar coordinates and we know that $\frac{\partial(r,\theta)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)} = 1$, this result should not come as a surprise.

$$\begin{aligned}
\frac{\partial(r,\theta)}{\partial(x,y)} &= \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix} \\
&= \begin{vmatrix} \frac{2x}{2\sqrt{x^2+y^2}} & \frac{2y}{2\sqrt{x^2+y^2}} \\ \frac{-y/x^2}{1+(y/x)^2} & \frac{1/x}{1+(y/x)^2} \end{vmatrix} \\
&= \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} \\
&= \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ -\frac{y}{r^2} & \frac{x}{r^2} \end{vmatrix} \\
&= \frac{x^2+y^2}{r^3} = \frac{1}{r}.
\end{aligned}$$

2. For the change of variables $x = u$, $y = \sqrt{r^2 - u^2}$, write $dx\,dy$ in terms of u and r.

Answer: We know $dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,r)}\right| du\,dr$.

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(u,r)} &= \begin{vmatrix} x_u & x_r \\ y_u & y_r \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 \\ \frac{-u}{\sqrt{r^2-u^2}} & \frac{r}{\sqrt{r^2-u^2}} \end{vmatrix} \\
&= \frac{r}{\sqrt{r^2-u^2}}
\end{aligned}$$

Hence $dx\,dy = \frac{r}{\sqrt{r^2-u^2}}\,du\,dr$.