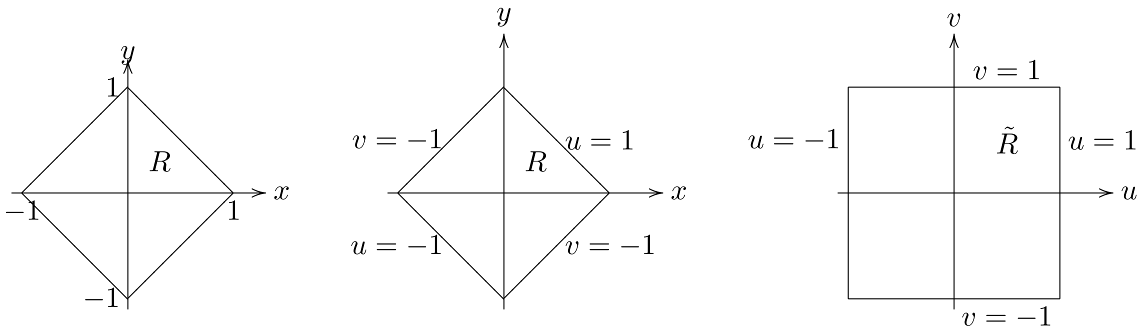
Problems: Change of Variables

Compute $\iint_R \left(\frac{x+y}{2-x+y}\right)^4 dx\,dy$, where R is the square with vertices at $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$.

Answer: Since the region is bounded by the lines $x+y=\pm 1$ and $x-y=\pm 1$, we make a change of variables:

$$u = x + y \quad v = x - y.$$



Computing the Jacobian: $\frac{\partial(u,v)}{\partial(x,y)} = -2 \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = -1/2.$

Thus, $dx\,dy = \frac{1}{2}\,du\,dv.$

Using either method 1 or method 2 we see the boundaries are given by $u = \pm 1$, $v = \pm 1 \Rightarrow$ the integral is $\iint_R \left(\frac{x+y}{w-x+y}\right)^4 dx\,dy = \int_{-1}^{1}\int_{-1}^{1}\left(\frac{u}{2-v}\right)^4 \frac{1}{2}\,du\,dv.$

Inner integral $= \left.\frac{u^5}{10(2-v)^4}\right|_{u=-1}^{u=1} = \frac{1}{5(2-v)^4}.$

Outer integral $= \left.\frac{1}{15(2-v)^3}\right|_{-1}^{1} = \frac{26}{405} \approx .06.$

We're integrating the fourth power of a fraction whose numerator ranges between -1 and 1 and whose denominator ranges between 1 and 3. The value of this integrand will always be positive and will often be small, so this answer seems reasonable.