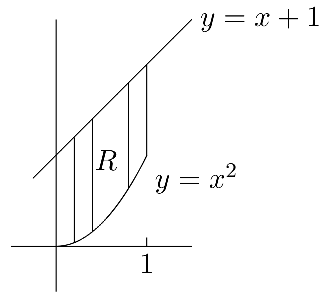
Problems: Regions of Integration

1. Find the mass of the region R bounded by
$y = x + 1$; $y = x^2$; $x = 0$ and $x = 1$, if density $= \delta(x, y) = xy$.



Answer:
Inner limits: y from x^2 to $x + 1$. Outer limits: x from 0 to 1.

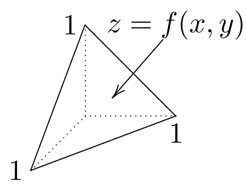
$$\Rightarrow M = \int\int_R \delta(x, y)\, dA = \int_{x=0}^{1} \int_{y=x^2}^{x+1} xy\, dy\, dx$$

Inner: $\displaystyle\int_{x^2}^{x+1} xy\, dy = x\frac{y^2}{2}\Big|_{x^2}^{x+1} = \frac{x(x+1)^2}{2} - \frac{x^5}{2} = \frac{x^3}{2} + x^2 + \frac{x}{2} - \frac{x^5}{2}.$

Outer: $\displaystyle\int_0^1 \frac{x^3}{2} + x^2 + \frac{x}{2} - \frac{x^5}{2}\, dx = \frac{x^4}{8} + \frac{x^3}{3} + \frac{x^2}{4} - \frac{x^6}{12}\Big|_0^1 = \frac{1}{8} + \frac{1}{3} + \frac{1}{4} - \frac{1}{12} = \frac{5}{8}.$

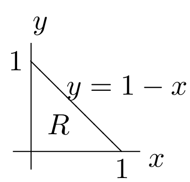
Note: The syntax $y = x^2$ in limits is redundant but useful. We know it must be y because of the dy matching the integral sign.

2. Find the volume of the tetrahedron shown below.



Tetrahedron

Answer: The surface has height: $z = 1 - x - y$.



Region R

Limits: inner: $0 < y < 1 - x$, outer: $0 < x < 1$. $\Rightarrow V = \displaystyle\int_{x=0}^{1} \int_{y=0}^{1-x} 1 - x - y\, dy\, dx.$

Inner: $\displaystyle\int_{y=0}^{1-x} 1 - x - y\, dy = y - xy - \frac{y^2}{2}\Big|_0^{1-x} = 1 - x - x + x^2 - \frac{1}{2} + x - \frac{x^2}{2}.$

Outer: $\displaystyle\int_0^1 \frac{1}{2} - x + \frac{x^2}{2}\, dx = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \frac{1}{6}.$