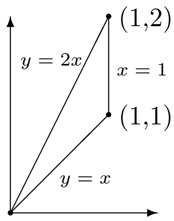
18.02 Exam 3 – Solutions

1. a)



b) $\int_0^1 \int_{y/2}^{y} dxdy + \int_1^2 \int_{y/2}^{1} dxdy.$

(the first integral corresponds to the bottom half $0 \le y \le 1$, the second integral to the top half $1 \le y \le 2$.)

2. a) $\delta dA = \frac{r\sin\theta}{r^2} r dr d\theta = \sin\theta dr d\theta.$

$$M = \iint_R \delta dA = \int_0^\pi \int_1^3 \sin\theta \; dr d\theta = \int_0^\pi 2\sin\theta d\theta = \big[-2\cos\theta\big]_0^\pi = 4.$$

b) $\bar{x} = \frac{1}{M}\iint_R x\delta dA = \frac{1}{4}\int_0^\pi \int_1^3 r\cos\theta\sin\theta dr d\theta$

The reason why one knows that $\bar{x} = 0$ without computation is that the region **and the density** are symmetric with respect to the y-axis ($\delta(x,y) = \delta(-x,y)$).

3. a) $N_x = -12y = M_y$, hence $\mathbf{F}$ is conservative.

b) $f_x = 3x^2 - 6y^2 \Rightarrow f = x^3 - 6y^2x + c(y) \Rightarrow f_y = -12xy + c'(y) = -12xy + 4y$. So $c'(y) = 4y$, thus $c(y) = 2y^2$ (+ constant). In conclusion

$$f = x^3 - 6xy^2 + 2y^2 \quad (+\text{ constant}).$$

c) The curve C starts at $(1,0)$ and ends at $(1,1)$, therefore

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(1,1) - f(1,0) = (1-6+2) - 1 = -4.$$

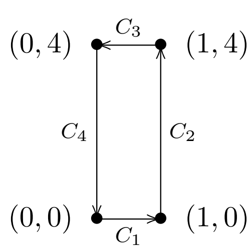
4. a) The parametrization of the circle C is $x = \cos t$, $y = \sin t$, for $0 \le t < 2\pi$; then $dx = -\sin t dt$, $dy = \cos t dt$ and

$$W = \int_0^{2\pi} (5\cos t + 3\sin t)(-\sin t)dt + (1 + \cos(\sin t))\cos t dt.$$

b) Let R be the unit disc inside C;

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R (N_x - M_y)dA = \iint_R (0-3)dA = -3 \text{ area}(R) = -3\pi.$$

5. a)



$$\oint_C \mathbf{F}\cdot\hat{\mathbf{n}}\, ds = \iint_R \operatorname{div}\mathbf{F}\, dxdy$$
$$= \iint_R (y + \cos x\cos y - \cos x\cos y)dxdy = \iint_R y dxdy$$
$$= \int_0^4 \int_0^1 y dxdy = \int_0^4 y dy = [y^2/2]_0^4 = 8.$$

b) On C_4, $x = 0$, so $\mathbf{F} = -\sin y\,\hat{\mathbf{j}}$, whereas $\hat{\mathbf{n}} = -\hat{\mathbf{i}}$. Hence $\mathbf{F} \cdot \hat{\mathbf{n}} = 0$. Therefore the flux of $\mathbf{F}$ through C_4 equals 0. Thus

$$\int_{C_1+C_2+C_3} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds - \int_{C_4} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds\ ;$$

and the total flux through $C_1 + C_2 + C_3$ is equal to the flux through C.

6. Let $u = 2x - y$ and $v = x + y - 1$. The Jacobian $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3$.

Hence $dudv = 3dxdy$ and $dxdy = \dfrac{1}{3}dudv$, so that

$$\begin{aligned}
V &= \iint\limits_{(2x-y)^2+(x+y-1)^2<4} (4 - (2x-y)^2 - (x+y-1)^2)\, dxdy \\
&= \iint\limits_{u^2+v^2<4} (4 - u^2 - v^2)\frac{1}{3}dudv \\
&= \int_0^{2\pi}\int_0^2 (4 - r^2)\frac{1}{3}rdrd\theta = \int_0^{2\pi}\left[\frac{2}{3}r^2 - \frac{1}{12}r^4\right]_0^2 d\theta \\
&= \int_0^{2\pi} \frac{4}{3}d\theta = \frac{8\pi}{3}.
\end{aligned}$$