Chain rule with more variables

1. Let $w = xyz$, $x = u^2 v$, $y = uv^2$, $z = u^2 + v^2$.

a) Use the chain rule to find $\dfrac{\partial w}{\partial u}$.

b) Find the total differential dw in terms of du and dv.

c) Find $\dfrac{\partial w}{\partial u}$ at the point $(u, v) = (1, 2)$.

Answer: a) The chain rule says

$$\begin{aligned}\frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \\ &= (yz)(2uv) + (xz)(v^2) + (xy)(2u).\end{aligned}$$

b) Using the formulas given we get

$$dw = yz\,dx + xz\,dy + xy\,dz$$

and

$$dx = 2uv\,du + u^2\,dv, \quad dy = v^2\,du + 2uv\,dv, \quad dz = 2u\,du + 2v\,dv.$$

Substituting for dx, dy, dz in the equation for dw gives

$$\begin{aligned}dw &= (yz)(2uv\,du + u^2\,dv) + (xz)(v^2\,du + 2uv\,dv) + (xy)(2u\,du + 2v\,dv). \\ &= (2yzuv + xzv^2 + 2xyu)\,du + (yzu^2 + 2xzuv + 2xyv)\,dv.\end{aligned}$$

Therefore

$$\frac{\partial w}{\partial u} = 2yzuv + xzv^2 + 2xyu \text{ and } \frac{\partial w}{\partial v} = yzu^2 + 2xzuv + 2xyv.$$

c) We do the chain of computations to compute the partial.

$$(u, v) = (1, 2) \;\Rightarrow\; (x, y, z) = (2, 4, 5) \;\Rightarrow\; \frac{\partial w}{\partial u} = (20)(4) + (10)(4) + (8)(2) = 136.$$