Chain Rule and Total Differentials

1. Find the total differential of $w = x^3yz + xy + z + 3$ at $(1, 2, 3)$.

Answer: The total differential at the point (x_0, y_0, z_0) is

$$dw = w_x(x_0, y_0, z_0)\,dx + w_y(x_0, y_0, z_0)\,dy + w_z(x_0, y_0, z_0)\,dz.$$

In our case,

$$w_x = 3x^2yz + y, \quad w_y = x^3z + x, \quad w_z = x^3y + 1.$$

Substituting in the point $(1, 2, 3)$ we get: $w_x(1, 2, 3) = 20,\ w_y(1, 2, 3) = 4,\ w_z(1, 2, 3) = 3.$
Thus,

$$dw = 20\,dx + 4\,dy + 3\,dz.$$

2. Suppose $w = x^3yz + xy + z + 3$ and

$$x = 3\cos t, \quad y = 3\sin t, \quad z = 2t.$$

Compute $\frac{dw}{dt}$ and evaluate it at $t = \pi/2$.

Answer: We do not substitute for x, y, z before differentiating, so we can practice the chain rule.

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\ &= (3x^2yz + y)(-3\sin t) + (x^3z + x)(3\cos t) + (x^3y + 1)(2). \end{aligned}$$

At $t = \pi/2$ we have $x = 0,\ y = 3,\ z = \pi, \qquad \sin\pi/2 = 1, \cos\pi/2 = 0.$
Thus,

$$\left.\frac{dw}{dt}\right|_{\pi/2} = 3(-3) + 3(0) + (1)2 = -7.$$

3. Show how the tangent approximation formula leads to the chain rule that was used in the previous problem.

Answer: The approximation formula is

$$\Delta w \approx \left.\frac{\partial f}{\partial x}\right|_o \Delta x + \left.\frac{\partial f}{\partial y}\right|_o \Delta y + \left.\frac{\partial f}{\partial z}\right|_o \Delta z.$$

If x, y, z are functions of time then dividing the approximation formula by Δt gives

$$\frac{\Delta w}{\Delta t} \approx \left.\frac{\partial f}{\partial x}\right|_o \frac{\Delta x}{\Delta t} + \left.\frac{\partial f}{\partial y}\right|_o \frac{\Delta y}{\Delta t} + \left.\frac{\partial f}{\partial z}\right|_o \frac{\Delta z}{\Delta t}.$$

In the limit as $\Delta t \to 0$ we get the chain rule.

Note: we use the regular 'd' for the derivative $\frac{dw}{dt}$ because in the chain of computations

$$t \to x, y, z \to w$$

the dependent variable w is ultimately a function of exactly one independent variable t. Thus, the derivative with respect to t is not a partial derivative.