Speed and arc length

1. A rocket follows a trajectory

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} = 10t\,\mathbf{i} + (-5t^2 + 10t)\mathbf{j}.$$

Find its speed and the arc length from $t = 0$ to $t = 1$.

<u>**Answer:**</u>

velocity $= \mathbf{v}(t) = \dfrac{d\mathbf{r}}{dt} = 10\mathbf{i}+(-10t+10)\mathbf{j} \;\Rightarrow\; \dfrac{ds}{dt} = \sqrt{10^2 + (-10t+10)^2} = 10\sqrt{1+(1-t)^2}.$

Arc length $L = \displaystyle\int_0^1 \frac{ds}{dt}\,dt = 10\int_0^1 \sqrt{1+(-t+1)^2}\,dt$

Make the change of variables $u = -t + 1$

$\Rightarrow\; du = -dt,\ t = 0 \to u = 1,\ t = 1 \to u = 0. \;\Rightarrow\; L = 10\displaystyle\int_0^1 \sqrt{1+u^2}\,du.$

We can compute this integral with the trig. substitution $u = \tan\theta$ or by use of tables

$\Rightarrow\; L = 10\displaystyle\int_0^{\pi/4} \sec^3\theta\,d\theta = 5\left[\sec\theta\,\tan\theta + \ln(\sec\theta + \tan\theta)\right|_0^{\pi/4} = 5(\sqrt{2} + \ln(\sqrt{2}+1)).$

2. For the cycloid $x = a\theta - a\sin\theta$, $y = a - a\cos\theta$ find the velocity, speed, unit tangent vector and arc length of one arch.

We will use the trigonometric formulas

$$\sin^2(\theta/2) = \frac{1-\cos\theta}{2} \quad \text{and} \quad \sin\theta = 2\sin(\theta/2)\cos(\theta/2).$$

Computing,

$$\frac{d\mathbf{r}}{d\theta} = a\langle 1-\cos\theta,\ \sin\theta\rangle = 2a\langle \sin^2(\theta/2),\ \sin(\theta/2)\cos(\theta/2)\rangle,$$

which implies

$$\left|\frac{d\mathbf{r}}{d\theta}\right| = \frac{ds}{d\theta} = 2a\sqrt{\sin^2(\theta/2)} = 2a|\sin(\theta/2)|.$$

So, $\mathbf{T} = \dfrac{2a\langle \sin^2(\theta/2),\ \sin(\theta/2)\cos(\theta/2)\rangle}{2a|\sin(\theta/2)|} = \pm\langle \sin(\theta/2),\ \cos(\theta/2)\rangle$ (a unit vector!)

Note, at the cusp ($\theta = 2\pi$) $ds/d\theta = 0$, i.e., you must stop to make a sudden 180 degree turn.

For one arch, $0 < \theta < 2\pi$, $\frac{ds}{d\theta} = 2a\sin(\theta/2)$

$$\begin{aligned} \Rightarrow \text{ arc length } &= \int_0^{2\pi} \frac{ds}{d\theta}\,d\theta \\ &= \int_0^{2\pi} 2a\sin(\theta/2)\,d\theta \\ &= -4a\cos(\theta/2)\Big|_0^{2\pi} \\ &= 8a \text{ (this is sometimes called Wren's theorem).} \end{aligned}$$