Product rule for vector derivatives

1. If $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are two parametric curves show the product rule for derivatives holds for the dot product.

Answer: This will follow from the usual product rule in single variable calculus. Lets assume the curves are in the plane. The proof would be exactly the same for curves in space. We want to prove that

$$\frac{d(\mathbf{r}_1 \cdot \mathbf{r}_2)}{dt} = \mathbf{r}_1' \cdot \mathbf{r}_2 \; + \; r_1 \cdot \mathbf{r}_2'.$$

Let $\mathbf{r}_1 = \langle x_1, y_1 \rangle$ and $\mathbf{r}_2 = \langle x_2, y_2 \rangle$. We have,

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = x_1 x_2 + y_1 y_2.$$

Taking derivatives using the product rule from single variable calculus, we get

$$\begin{aligned}
\frac{d(\mathbf{r}_1 \cdot \mathbf{r}_2)}{dt} &= \frac{d(x_1 x_2 + y_1 y_2)}{dt} \\
&= x_1' x_2 + x_1 x_2' + y_1' y_2 + y_1 y_2' \\
&= (x_1' x_2 + y_1' y_2) + (x_1 x_2' + y_1 y_2') \\
&= \langle x_1', y_1' \rangle \cdot \langle x_2, y_2 \rangle + \langle x_1, y_1 \rangle \cdot \langle x_2', y_2' \rangle \\
&= \mathbf{r}_1' \cdot \mathbf{r}_2 \; + \; \mathbf{r}_1 \cdot \mathbf{r}_2'. \quad \blacksquare
\end{aligned}$$