Product rule for vector derivatives

1. If $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are two parametric curves show the product rule for derivatives holds for the cross product.

Answer: As with the dot product, this will follow from the usual product rule in single variable calculus. We want to show

$$\frac{d(\mathbf{r}_1 \times \mathbf{r}_2)}{dt} = \mathbf{r}_1' \times \mathbf{r}_2 \; + \; r_1 \times \mathbf{r}_2'.$$

Let $\mathbf{r}_1 = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{r}_2 = \langle x_2, y_2, z_2 \rangle$. We have,

$$\mathbf{r}_1 \times \mathbf{r}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \langle y_1 z_2 - z_1 y_2,\; z_1 x_2 - x_1 z_2,\; x_1 y_2 - y_1 x_2 \rangle.$$

Taking derivatives using the product rule from single variable calculus, we get a lot of terms, which we can group to prove the vector formula.

$$\begin{aligned} \frac{d(\mathbf{r}_1 \times \mathbf{r}_2)}{dt} &= \langle y_1' z_2 + y_1 z_2' - z_1' y_2 - z_1 y_2',\; z_1' x_2 + z_1 x_2' - x_1' z_2 - x_1 z_2',\; x_1' y_2 + x_1 y_2' - y_1' x_2 - y_1 x_2' \rangle \\ &= \langle (y_1' z_2 - z_1' y_2) + (y_1 z_2' - z_1 y_2'),\; (z_1' x_2 - x_1' z_2) + (z_1 x_2' - x_1 z_2'),\; (x_1' y_2 - y_1' x_2) + (x_1 y_2' - y_1 x_2') \rangle \\ &= \langle x_1', y_1', z_1' \rangle \times \langle x_2, y_2, z_2 \rangle + \langle x_1, y_1, z_1 \rangle \times \langle x_2', y_2', z_2' \rangle \\ &= \mathbf{r}_1' \times \mathbf{r}_2 \; + \; \mathbf{r}_1 \times \mathbf{r}_2'. \quad \blacksquare \end{aligned}$$