Solutions to linear systems

1. Consider the system of equations

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 1 \\ 4x & + & 5y & + & 6z & = & 2 \\ 7x & + & 8y & + & cz & = & 3. \end{array}$$

a) Write the system in matrix form.

b) For which values of c is there exactly one solution?

c) For which values of c are there either 0 or infinitely many solutions?

d) Take the corresponding homogeneous system

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 0 \\ 4x & + & 5y & + & 6z & = & 0 \\ 7x & + & 8y & + & cz & = & 0. \end{array}$$

For the value(s) of c found in part (c) give *all* the solutions.

Answer: a) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

b) There is exactly one solution when the coefficient matrix has an inverse (i.e., is *invertible*). This happens when the determinant is not zero.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & c \end{vmatrix} = 1(5c-48) - 2(4c-42) + 3(32-35) = -3c + 27 = 0 \Leftrightarrow c = 9.$$

There is exactly one solution as long as $c \neq 9$.

c) This is just the complement of part (b): there are zero or infinitely many solutions when $c = 9$.

d) Setting $c = 9$ our coefficient matrix is $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. Thinking of matrix multiplication as a series of dot products between rows of the left matrix and column(s) of the right one we see that in solving

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

we are looking for vectors $\langle x, y, z \rangle$ that are orthogonal to each of the rows of A. Since $\det(A) = 0$, the rows are all in a plane and we can find orthogonal vectors by taking a cross product of (say) the first two rows.

$$\langle 1, 2, 3 \rangle \times \langle 4, 5, 6 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \langle -3, 6, -3 \rangle.$$

Since scaling will preserve orthogonality, all the solutions are scalar multiples, i.e., all the solutions are of the form $(x, y, z) = (-3a, 6a, -3a)$. We can make this a little nicer by removing the common factor of three,

$$(x, y, z) = (-a, 2a, -a) = a(-1, 2, -1).$$