Matrix inverses

1. a) Find the inverse of $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 4 & 0 \\ 2 & 1 & 5 \end{pmatrix}$.

b) Use part (a) to solve the system of equations

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & 1 \\ x & + & 4y & & & = & 0 \\ 2x & + & y & + & 5z & = & 3 \end{array}$$

Answer: a) We compute in sequence: the determinant, the matrix of minors, the matrix of cofactors, the adjoint matrix, the inverse. Note, in computing the determinant by Laplace expansion we compute some minors which we use in the matrix of minors.

$|A| = 1(20)-2(5)+1(-7) = 3$; minors $= \begin{pmatrix} 20 & 5 & -7 \\ 9 & 3 & -3 \\ -4 & -1 & 2 \end{pmatrix}$; cofactors $= \begin{pmatrix} 20 & -5 & -7 \\ -9 & 3 & 3 \\ -4 & 1 & 2 \end{pmatrix}$;

adjoint $= \begin{pmatrix} 20 & -9 & -4 \\ -5 & 3 & 1 \\ -7 & 3 & 2 \end{pmatrix}$; $A^{-1} = \frac{1}{3}\begin{pmatrix} 20 & -9 & -4 \\ -5 & 3 & 1 \\ -7 & 3 & 2 \end{pmatrix}$.

b) In matrix form the system is

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

Solving using the inverse we get

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 20 & -9 & -4 \\ -5 & 3 & 1 \\ -7 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 8/3 \\ -2/3 \\ -1/3 \end{pmatrix}.$$

2. a) Find $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$ using the method of cofactors.

b) Memorize these steps for finding the inverse of a 2×2 matrix:
(i) Switch a and d. (ii) Change the signs on b and c. (iii) Divide by the determinant.

c) Find $\begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix}^{-1}$.

Answer: a) determinant $= ad - bc$; minors $= \begin{pmatrix} d & c \\ b & a \end{pmatrix}$; cofactors $= \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$;

adjoint $= \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$; inverse $= \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

b) Done. (Good memorizing!)

c) $\begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{7}\begin{pmatrix} 2 & -5 \\ -1 & 6 \end{pmatrix}$.