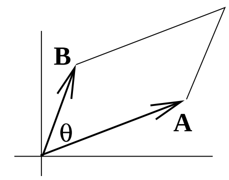
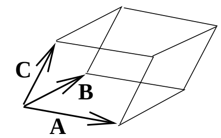
Determinants 2. Area and Volume



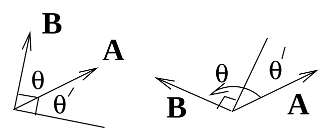
Area and volume interpretation of the determinant:

(1) $\quad \pm\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} =$ area of parallelogram with edges $\mathbf{A} = (a_1, a_2),\ \mathbf{B} = (b_1, b_2)$.



(2) $\quad \pm\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$ volume of parallelepiped with edges row-vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

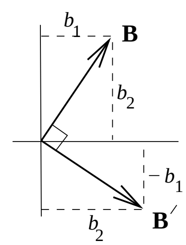
In each case, choose the sign which makes the left side non-negative.



Proof of (1). We begin with two preliminary observations.

Let θ be the positive angle from $\mathbf{A}$ to $\mathbf{B}$; we assume it is $< \pi$, so that $\mathbf{A}$ and $\mathbf{B}$ have the general positions illustrated.

Let $\theta' = \pi/2 - \theta$, as illustrated. Then $\cos\theta' = \sin\theta$.



Draw the vector $\mathbf{B}'$ obtained by rotating $\mathbf{B}$ to the right by $\pi/2$. The picture shows that $\mathbf{B}' = (b_2, -b_1)$, and $|\mathbf{B}'| = |\mathbf{B}|$.

To prove (1) now, we have a standard formula of Euclidean geometry,

$$\begin{aligned} \text{area of parallelogram} &= |\mathbf{A}||\mathbf{B}|\sin\theta \\ &= |\mathbf{A}||\mathbf{B}'|\cos\theta', && \text{by the above observations} \\ &= \mathbf{A}\cdot\mathbf{B}', && \text{by the geometric definition of dot product} \\ &= a_1 b_2 - a_2 b_1 && \text{by the formula for } \mathbf{B}' \end{aligned}$$

This proves the area interpretation (1) if $\mathbf{A}$ and $\mathbf{B}$ have the position shown. If their positions are reversed, then the area is the same, but the sign of the determinant is changed, so the formula has to read,

$$\text{area of parallelogram} = \pm\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \quad \text{whichever sign makes the right side } \geq 0.$$

The proof of the analogous volume formula (2) will be made when we study the scalar triple product $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$.

Generalizing (1) and (2), $n \times n$ determinants can be interpreted as the hypervolume in n-space of a n-dimensional parallelotope.