

Determinants 1.

Given a square array A of numbers, we associate with it a number called the **determinant** of A , and written either $\det(A)$, or $|A|$. For 2×2

$$(1) \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Do not memorize this as a formula — learn instead the pattern which gives the terms. The 2×2 case is easy: the product of the elements on one diagonal (the “main diagonal”), minus the product of the elements on the other (the “antidiagonal”).

Below we will see how to compute 3×3 determinants $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$. First, try the following 2×2 example on your own, then check your work against the solution.

Example 1.1 Evaluate $\begin{vmatrix} 1 & -2 \\ -1 & 3 \end{vmatrix}$ using (1).

Solution. Using the same order as in (1), we get $12 + (-8) + 1 - 6 - 8 - (-2) = -7$.

Important facts about $|A|$:

- D-1.** $|A|$ is multiplied by -1 if we interchange two rows or two columns.
- D-2.** $|A| = 0$ if one row or column is all zero, or if two rows or two columns are the same.
- D-3.** $|A|$ is multiplied by c , if every element of some row or column is multiplied by c .
- D-4.** The value of $|A|$ is unchanged if we add to one row (or column) a constant multiple of another row (resp. column).

All of these facts are easy to check for 2×2 determinants from the formula (1); from this, their truth also for 3×3 determinants will follow from the Laplace expansion.

Though the letters a, b, c, \dots can be used for very small determinants, they can't for larger ones; it's important early on to get used to the standard notation for the entries of determinants. This is what the common software packages and the literature use. The determinants of order two and three would be written respectively

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

In general, the **ij-entry**, written a_{ij} , is the number in the i -th row and j -th column.

Its **ij-minor**, written $|A_{ij}|$, is the determinant that's left after deleting from $|A|$ the row and column containing a_{ij} .

Its **ij-cofactor**, written here A_{ij} , is given as a formula by $A_{ij} = (-1)^{i+j}|A_{ij}|$. For a 3×3 determinant, it is easier to think of it this way: we put $+$ or $-$ in front of the ij -minor, according to whether $+$ or $-$ occurs in the ij -position in the checkerboard pattern

$$(2) \quad \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}.$$

Example 1.2 $|A| = \begin{vmatrix} 1 & 0 & 3 \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix}$. Find $|A_{12}|$, A_{12} , $|A_{22}|$, A_{22} .

Solution. $|A_{12}| = \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = 1$, $A_{12} = -1$. $|A_{22}| = \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} = -7$, $A_{22} = -7$.

Laplace expansion by cofactors

This is another way to evaluate a determinant; we give the rule for a 3×3 . It generalizes easily to an $n \times n$ determinant.

Select any row (or column) of the determinant. Multiply each entry a_{ij} in that row (or column) by its cofactor A_{ij} , and add the three resulting numbers; you get the value of the determinant.

As practice with notation, here is the formula for the Laplace expansion of a third order (i.e., a 3×3) determinant using the cofactors of the first row:

$$(3) \quad a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = |A|$$

and the formula using the cofactors of the j -th column:

$$(4) \quad a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j} = |A|$$

Example 1.3 Evaluate the determinant in Example 1.2 using the Laplace expansions by the first row and by the second column, and check by also using (1).

Solution. The Laplace expansion by the first row is

$$\begin{vmatrix} 1 & 0 & 3 \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} + 3 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 \cdot (-1) - 0 \cdot 1 + 3 \cdot (-3) = -10.$$

The Laplace expansion by the second column would be

$$\begin{vmatrix} 1 & 0 & 3 \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix} = -0 \cdot \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} = 0 + 2 \cdot (-7) - 1 \cdot (-4) = -10.$$

Checking by (1), we have $|A| = -2 + 0 + 3 - 12 - 0 - (-1) = -10$.

Example 1.4 Show the Laplace expansion by the first row gives the following formula (which you may have seen before).

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + chd - gec - hfa - ibd$$

Solution. We have

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ei - fh) - b(di - fg) + c(dh - eg), \end{aligned}$$

whose six terms agree with the six terms on the right of the formula above.

(A similar argument can be made for the Laplace expansion by any row or column.)

For $n \times n$ determinants, the **minor** $|A_{ij}|$ of the entry a_{ij} is defined to be the determinant obtained by deleting the i -th row and j -th column; the **cofactor** A_{ij} is the minor, prefixed by a $+$ or $-$ sign according to the natural generalization of the checkerboard pattern (2). Then the Laplace expansion by the i -th row would be

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in} .$$

This is an inductive calculation — it expresses the determinant of order n in terms of determinants of order $n - 1$. Thus, since we can calculate determinants of order 3, it allows us to calculate determinants of order 4; then determinants of order 5, and so on. If we take for definiteness $i = 1$, then the above Laplace expansion formula can be used as the basis of an inductive definition of the $n \times n$ determinant.

Example 1.5 Evaluate $\begin{vmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 1 & 4 \\ -1 & 4 & 1 & 0 \\ 0 & 4 & 2 & -1 \end{vmatrix}$ by its Laplace expansion by the first row.

Solution. $1 \cdot \begin{vmatrix} -1 & 1 & 4 \\ 4 & 1 & 0 \\ 4 & 2 & -1 \end{vmatrix} - 0 \cdot A_{12} + 2 \cdot \begin{vmatrix} 2 & -1 & 4 \\ -1 & 4 & 0 \\ 0 & 4 & -1 \end{vmatrix} - 3 \cdot \begin{vmatrix} 2 & -1 & 1 \\ -1 & 4 & 1 \\ 0 & 4 & 2 \end{vmatrix}$
 $= 1 \cdot 21 + 2 \cdot (-23) - 3 \cdot 2 = -31.$

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