18.02 Notes on Divergence and Partial Differential Equations

This section describes the role played by the divergence theorem in the study of heat flow and motion in liquids and gases.[1]

We will illustrate using the example of smoke in the air. The same principles apply to a dye in water or a drug in the bloodstream. We distinguish two factors contributing to the motion of the smoke. The first is called *diffusion*, the spreading of the smoke, and the second is called *convection*, the transport of the smoke by the wind.

The smoke is modeled by a function $u(t, x, y, z)$ that represents the density or concentration of smoke. For example, if we assume that all the smoke particles are of equal size, then we can define u as the number of smoke particles per cubic centimeter. The quantity u depends the variables representing time t and space (x, y, z). Suppose that the movement of particles of smoke is represented by a flow rate $\mathbf{F}$. The starting place is the equation

$$\frac{d}{dt}\iiint_D u\,dV = -\iint_S \mathbf{F}\cdot d\mathbf{S} \tag{1}$$

for any surface S enclosing a region D. This equation says that the net rate of change of the amount of u in D equals minus the amount of u that flows out through S (flux of $\mathbf{F}$ through S). The minus sign comes from the convention that $d\mathbf{S} = \mathbf{n}dS$ is chosen so that $\mathbf{n}$ points outward, so that the flux is positive when smoke is flowing out through S.

Formula (1) is the integral form of an equation that has an equivalent differential, or infinitesimal, form given by

$$\frac{\partial u}{\partial t} = -\text{div }\mathbf{F} \tag{2}$$

The differential form is easier to deal with than the integral form for both practical and theoretical reasons.

The divergence theorem is used to show that (1) and (2) are equivalent, as follows. First, to see that (2) implies (1), integrate (2) over the region D, then apply the divergence theorem,

$$\iiint_D \frac{\partial u}{\partial t}\,dV = \iiint_D (-\text{div }\mathbf{F})\,dV = -\iint_S \mathbf{F}\cdot d\mathbf{S} \tag{3}$$

Rewrite the left-hand side of (1) by exchanging the order of differentiation and integration. Thus

$$\frac{d}{dt}\iiint_D u\,dV = \iiint_D \frac{\partial u}{\partial t}\,dV \tag{4}$$

(Differentiation under the integral sign was explained in an earlier note. It is analogous to differentiating a sum term by term.) Combining (3) with (4) yields (1).

Conversely, if one starts with (1), then applying the divergence theorem to the right-hand side of (1), one obtains

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_D \text{div }\mathbf{F}\,dV \tag{5}$$

Combining (1), (4), and (5) gives

$$\iiint_D \frac{\partial u}{\partial t}\, dV = -\iiint_D \text{div } \mathbf{F}\, dV \tag{6}$$

To deduce the infinitesimal version of (6), namely (2), the trick is to take the average value. Dividing (6) by vol (D),

$$\frac{1}{\text{vol}\,(D)} \iiint_D \frac{\partial u}{\partial t}\, dV = \frac{1}{\text{vol}\,(D)} \iiint_D (-\text{div } \mathbf{F})\, dV \tag{6$'$}$$

Now take the limit in ($6'$) as D shrinks to a point P_0. The value of each side approaches the value of the integrand at P_0, so at each point P_0

$$\frac{\partial u}{\partial t} = -\text{div } \mathbf{F}$$

In other words, the partial differential equation (2) is valid.

Example 1. The diffusion equation. In the case of diffusion alone, that is, when there is no motion of the air, the smoke spreads slowly and lazily according to the formula

$$\mathbf{F} = -k\nabla u$$

To explain this physical law, consider a screen or membrane and the net flow of particles across the screen. The flow is from higher concentration to lower concentration, so it points in the direction of $-\nabla u$. The flow rate, the magnitude $|\mathbf{F}|$, is faster when the difference in concentrations is greater. The simplest such relationship is ordinary proportionality: $|\mathbf{F}| = k|\nabla u|$. Together these formulas for the direction and magnitude of $\mathbf{F}$ give $\mathbf{F} = -k\nabla u$.

Combined $\mathbf{F} = -k\nabla u$ with (2) one finds

$$\frac{\partial u}{\partial t} = -\text{div } \mathbf{F} = -\text{div } (-k\nabla u) = k\nabla^2 u$$

In other words,

$$\frac{\partial u}{\partial t} = k\nabla^2 u \qquad \left(= k\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u\right) \tag{7}$$

This is known as the diffusion equation. It is also known as the *heat equation* because it is also satisfied when u is interpreted as a temperature rather than as a concentration of a dye or of smoke.

Example 2. Convection or Advection. Not only does smoke diffuse in the air it is also carried by the wind. The motion of a velocity $\mathbf{v}$ is modeled by

$$\mathbf{F} = u\mathbf{v} \tag{8}$$

This expresses the idea that the entire quantity of u at a given position is moved by the velocity vector $\mathbf{v}$. The convection equation is then

$$\frac{\partial u}{\partial t} = -\text{div } (u\mathbf{v}) \tag{9}$$

Next, let us us show how this convection or transport equation reinforces our physical intuition for what the notion of divergence means. Consider the case of constant density $u = c$ at time $t = 0$. When we say that a fluid, such a water, is incompressible, we mean that the volume of fluid does not change with time. Put another way, if the density u starts out constant then it should remain constant for all time. For $u = c$, $\partial u/\partial t = 0$. It follows from (9) that

$$\text{div } \mathbf{v} = \mathbf{0} \tag{10}$$

A flow $\mathbf{v}$ satisfying (10) is known as *incompressible*. Although the word refers only to compression not expansion, its physical meaning is that the fluid neither expands nor contracts.

For an incompressible flow $\mathbf{v}$, equation (9) simplifies, because by the product rule

$$\text{div } (u\mathbf{v}) = \nabla u \cdot \mathbf{v} + u \, \text{div } \mathbf{v} = \mathbf{v} \cdot \nabla u$$

The transport (or convection) equation for incompressible flow takes the more familiar form

$$\frac{\partial u}{\partial t} = -\mathbf{v} \cdot \nabla u \quad \Longleftrightarrow \quad \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) u = 0 \tag{9'}$$

Example 3. Convection-diffusion. Combining the effects of diffusion with convection yields the equation governing the motion of smoke (and also of heat in a traditional convection oven). To do this, just add together diffusion and convection:

$$\mathbf{F} = -k\nabla u + u\mathbf{v} \quad \Longleftrightarrow \quad \frac{\partial u}{\partial t} = -\text{div } \mathbf{F} = k\nabla^2 u - \text{div } (u\mathbf{v}) \quad \Longleftrightarrow \quad \frac{\partial u}{\partial t} + \text{div } (u\mathbf{v}) = k\nabla^2 u$$

The convection-diffusion equation (also called advection-diffusion) governs the concentration of smoke u blowing in the wind with given velocity $\mathbf{v}$. A traditional oven is called a convection oven because the heat is transported near the food by the circulation of the air. (When the oven operates on the broiler setting, however, radiant heat does most of the cooking.)

Example 4. Navier-Stokes Equation.

To determine the velocity field $\mathbf{v}$ of the air, a further system of equations is required. These equations are known as the Navier-Stokes equations. In the incompressible case, they are

$$\text{div } (\mathbf{v}) = 0; \quad \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) \mathbf{v} = \nu\nabla^2\mathbf{v} - \nabla p$$

Here ν is a physical constant known as the viscosity. It plays the same mathematical role as the diffusion contant k for smoke. The letter p denotes a scalar function intepreted physically as pressure. The Navier-Stokes equation says that $\mathbf{v}$ itself satisfies a convection-diffusion equation with an extra pressure term.