18.02 - Solutions of Practice Final A - Spring 2006

Problem 1. $\overrightarrow{PQ} = \langle 2, 0, 3\rangle$; $\overrightarrow{PR} = \langle 1, -2, 2\rangle$; $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 2 & 0 & 3 \\ 1 & -2 & 2 \end{vmatrix} = 6\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} - 4\hat{\boldsymbol{k}}$

Equation of the plane: $6x - y - 4z = d$. Plane passing through P: $6 \cdot 0 - 1 - 4 \cdot 0 = d$.
Equation of the plane: $6x - y - 4z = -1$.

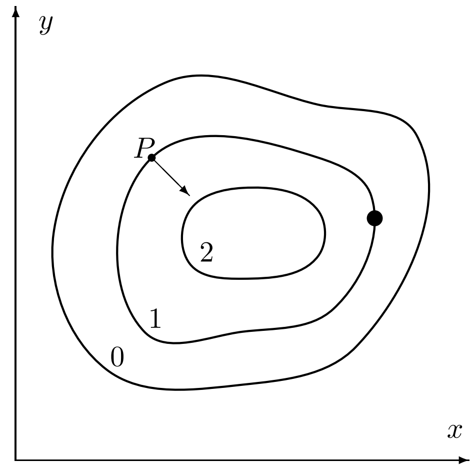
Problem 2. Parametric equation for the line: $P_1 + t\overrightarrow{P_1P_2} = (-1, 2, -1) + t\langle 2, 2, 1\rangle =$
$= (-1 + 2t, 2 + 2t, -1 + t)$, that is $x(t) = -1 + 2t$, $y(t) = 2 + 2t$, $z(t) = -1 + t$.
Intersection: $3x(t) - 2y(t) + z(t) = 1 \implies -3 + 6t - 4 - 4t - 1 + t = 1 \implies -8 + 3t = 1$, that is $t = 3$, which corresponds to the point $(5, 8, 2)$.
The function $3x - 2y + z - 1$ takes value -1 at the origin and -6 at P_2, which are both negative. So P_2 and the origin are in the same half-space.

Problem 3. a) A is not invertible if and only if $\det(A) = 0$.
$\det(A) = 1\begin{vmatrix} 4 & c \\ c & 2 \end{vmatrix} - 2\begin{vmatrix} -1 & c \\ 3 & 2 \end{vmatrix} + \begin{vmatrix} -1 & 4 \\ 3 & c \end{vmatrix} = (8 - c^2) - 2(-2 - 3c) + (-c - 12) = -c^2 + 5c =$
$= c(5 - c)$, hence A is not invertible if and only if $c = 0$ or $c = 5$.
b) For $c = 1$, $\det(A) = 4$.
If $A^{-1} = \begin{pmatrix} . & . & . \\ . & . & a \\ . & . & b \end{pmatrix}$, then $a = -\frac{1}{4}\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = -\frac{1}{2}$ and $b = \frac{1}{4}\begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = \frac{3}{2}$.

Problem 4. a) $\vec{\boldsymbol{v}}(t) = e^t\langle \cos t - \sin t, \sin t + \cos t\rangle$ and $|\vec{\boldsymbol{v}}(t)|^2 = e^{2t}(\cos^2 t + \sin^2 t - 2\sin t\cos t +$
$+ \sin^2 t + \cos^2 t + 2\sin t\cos t) = 2e^{2t}$, so the speed is $|\vec{\boldsymbol{v}}(t)| = \sqrt{2}e^t$.
b) $\cos\theta = \dfrac{\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{v}}}{|\vec{\boldsymbol{r}}|\,|\vec{\boldsymbol{v}}|} = \dfrac{e^{2t}\langle \cos t, \sin t\rangle \cdot \langle \cos t - \sin t, \sin t + \cos t\rangle}{\sqrt{2}e^{2t}} = \dfrac{\sqrt{2}}{2}$, so $\theta = \pm\pi/4$.

Problem 5. a) $\nabla f = \langle 3x^2 + y^2, 2xy - 2\rangle$ and $\nabla f(1, 2) = \langle 7, 2\rangle$.
$f(1.1, 1.9) \approx f(1, 2) + \langle 0.1, -0.1\rangle \cdot \nabla f(1, 2) = 1 + 0.7 - 0.2 = 1.5$.
b) The velocity is $\vec{\boldsymbol{v}}(t) = \langle 3t^2, 4t\rangle$ and $\vec{\boldsymbol{v}}(1) = \langle 3, 4\rangle$.
$t = 1$ corresponds to the point $(1, 2)$, so
$\dfrac{df}{dt}(1) = \dfrac{\partial f}{\partial x}(1, 2)\dfrac{dx}{dt}(1) + \dfrac{\partial f}{\partial y}(1, 2)\dfrac{dy}{dt}(1) = 7 \cdot 3 + 2 \cdot 4 = 29.$

Problem 6.



Problem 7. a) $\nabla f = \langle 3x^2 - y, -x + y\rangle$.

Critical points: $\nabla f = 0 \iff \begin{cases} y = 3x^2 \\ x = y \end{cases}$

The critical points are $(0,0)$ and $(1/3, 1/3)$.

b) $f_{xx} = 6x$, $f_{xy} = -1$, $f_{yy} = 1$, so $\Delta = 6x - 1$. At the origin $\Delta(0,0) = -1 < 0$, so it is a saddle point.

c) On the boundary $x = 0$ and $f(0,y) = y^2/2$, so the minimum at the boundary is 0 attained at $(0,0)$. The maximum value is $+\infty$.

$f(x,y) = x^3 - \dfrac{x^2}{2} + \dfrac{1}{2}(y-x)^2$, so $f(x,y) \to +\infty$ for $x \to +\infty$ and/or $y \to \pm\infty$. Hence the minimum can be either at $(0,0)$ or at $(1/3,1/3)$. Because $f(1/3,1/3) = -1/54$, this is the minimum value.

Problem 8. a) Let $g(x,y,z) = x^3 + yz - 1$. Then $\nabla g = \langle 3x^2, z, y\rangle$ and $\nabla g(-1,2,1) = \langle 3,1,2\rangle$, hence the equation of the tangent plane is $3x + y + 2z = d$.
It must pass through $(-1,2,1)$, so $3(-1) + 2 + 2(1) = d \implies d = 1$.
Equation of the tangent plane: $3x + y + 2z = 1$.

b) Constraint $\implies 3dx + dy + 2dz = 0$ at $(-1,2,1)$. Keeping z fixed, we get $dx = -dy/3$. Because $df = a\,dx + b\,dy + c\,dz$ at $(-1,2,1)$, we obtain $df = (-a/3 + b)dy$, that is $\left(\dfrac{\partial f}{\partial y}\right)_z (-1,2,1) = b - \dfrac{a}{3}$.

Problem 9. $\displaystyle\int_0^1 \int_0^{\sqrt{x}} \frac{2xy}{1-y^4}\,dy\,dx = \int_0^1 \int_{y^2}^1 \frac{2xy}{1-y^4}\,dx\,dy = \int_0^1 \frac{y}{1-y^4}\Big[x^2\Big]_{x=y^2}^{x=1} dy =$

$\displaystyle = \int_0^1 y\,dy = 1/2.$

Problem 10. *Direct method.* The circle is parametrized by $x(\theta) = a\cos\theta$, $y(\theta) = a\sin\theta$, for $0 \le \theta \le 2\pi$. The work is $\displaystyle\int_C \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{r}} = \int_C -y^3 dx + x^3 dy =$

$$= \int_0^{2\pi} -a^3\sin^3\theta(-a\sin\theta\,d\theta) + a^3\cos^3\theta(a\cos\theta\,d\theta) = a^4 \int_0^{2\pi} (\sin^4\theta + \cos^4\theta)d\theta =$$

$= 8a^4 \int_0^{\pi/2} \sin^4\theta\, d\theta =$ (using the table) $= \frac{3\pi}{2} a^4$.

Using Green's theorem. $\int_C \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{r}} = \iint_R (N_x - M_y) dA$, where R is the disc of radius a, $M = -y^3$ and $N = x^3$, so that $N_x - M_y = 3x^2 + 3y^2 = 3r^2$.

Hence the work is $\int_0^{2\pi} \int_0^a 3r^2 \cdot r\, dr\, d\theta = \int_0^{2\pi} d\theta \left[\frac{3r^4}{4}\right]_0^a = \frac{3\pi}{2} a^4$.

Problem 11. Call $\vec{\mathbf{F}} = x\hat{\boldsymbol{\imath}}$ and recall that (Flux) $= \int_C \vec{\mathbf{F}} \cdot \hat{\mathbf{n}}\, ds$.

Side $x = -1$: $\hat{\mathbf{n}} = -\hat{\boldsymbol{\imath}}$, $\vec{\mathbf{F}} \cdot \hat{\mathbf{n}} = 1$, so the flux is 2.
Side $x = 1$: $\hat{\mathbf{n}} = \hat{\boldsymbol{\imath}}$, $\vec{\mathbf{F}} \cdot \hat{\mathbf{n}} = 1$, so the flux is 2.
Side $y = -1$: $\hat{\mathbf{n}} = -\hat{\boldsymbol{\jmath}}$, $\vec{\mathbf{F}} \cdot \hat{\mathbf{n}} = 0$, so the flux is 0.
Side $y = 1$: $\hat{\mathbf{n}} = \hat{\boldsymbol{\jmath}}$, $\vec{\mathbf{F}} \cdot \hat{\mathbf{n}} = 0$, so the flux is 0.
The total flux out of any square S of sidelength 2 is always 4, because Green's theorem in normal form says it is equal to $\iint_S (M_x + N_y) dA = \iint_S 1 \cdot dA = \text{Area}(S) = 2^2 = 4$.

Problem 12. Green's theorem in normal form: $\int_C \vec{\mathbf{F}} \cdot \hat{\mathbf{n}}\, ds = \iint_R \text{div}(\vec{\mathbf{F}}) dA$, where R is the region enclosed by C.

$\text{div}(\vec{\mathbf{F}}) = 2x - y + 2$, so the flux is given by $\iint_{(2x-y)^2+(5x-y)^2<3} (2x - y + 2)\, dx\, dy$.

Change of variables: $u = 2x - y$, $v = 5x - y$, so

$$dx\, dy = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} du\, dv = \left|\det\begin{pmatrix} 2 & -1 \\ 5 & -1 \end{pmatrix}\right|^{-1} du\, dv = \frac{1}{3}\, du\, dv.$$

The integral becomes $\iint_{u^2+v^2<3} \frac{u+2}{3}\, du\, dv$. Using the symmetry $(u,v) \mapsto (-u,v)$, we have that the integral $\iint_{u^2+v^2<3} \frac{u}{3}\, du\, dv = 0$, so that the flux is given by
$\iint_{u^2+v^2<3} \frac{2}{3}\, du\, dv = \frac{2}{3}\pi(\sqrt{3})^2 = 2\pi$.

Problem 13. In cylindrical coordinates the volume is $\int_0^a \int_0^{2\pi} \int_0^1 r\, dr\, d\theta\, dz$.

In spherical coordinates $\int_0^{2\pi} \int_0^{\arctan(1/a)} \int_0^{a/\cos\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta +$
$+ \int_0^{2\pi} \int_{\arctan(1/a)}^{\pi/2} \int_0^{1/\sin\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$.

Problem 14. a) $\vec{\mathbf{F}}$ is conservative if and only if $\vec{\nabla} \times \vec{\mathbf{F}} = 0$ (because $\vec{\mathbf{F}}$ is continuous and differentiable everywhere).

$\overrightarrow{\nabla} \times \overrightarrow{\mathbf{F}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \partial_x & \partial_y & \partial_z \\ z^2 & z \sin y & 2z + axz + b\cos y \end{vmatrix} = (-b\sin y - \sin y)\hat{\boldsymbol{\imath}} - (az - 2z)\hat{\boldsymbol{\jmath}}$, so we must have $a = 2$ and $b = -1$.

b) Let $\overrightarrow{\mathbf{F}} = \nabla f$. We must have $f_z = 2z + 2xz - \cos y$, so $f(x, y, z) = z^2 + xz^2 - z\cos y + g(x, y)$. Moreover, $z\sin y + g_y(x, y) = f_y = z\sin y \implies g(x, y) = h(x)$. Finally, $z^2 + h'(x) = z^2 \implies h(x) = \text{constant}$. Hence, $f(x, y, z) = z^2 + xz^2 - z\cos y$ is a potential for $\overrightarrow{\mathbf{F}}$.

c) The curve goes from $(-1, 0, -1)$ to $(1, 0, 1)$. Fundamental theorem of calculus for line integrals: $\int_C \overrightarrow{\mathbf{F}} \cdot d\vec{\boldsymbol{r}} = f(1, 0, 1) - f(-1, 0, -1) = 1 - 1 = 0$.

Problem 15. *Direct method.* On the xy-plane, $\hat{\mathbf{n}} = -\hat{\boldsymbol{k}}$, $\overrightarrow{\mathbf{F}} \cdot \hat{\mathbf{n}} = -1$, so the flux is $\pi(2)^2 = -4\pi$. On the portion S of paraboloid, we compute $\iint_S \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{S}}$ by integrating over the shadow of S in the xy-plane.

$d\overrightarrow{\mathbf{S}} = \langle 2x, 2y, 1\rangle\, dx\, dy$, so $\overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{S}} = (2x^2 + 2y^2 + 1 - 2z)\, dx\, dy =$
$= [2x^2 + 2y^2 + 1 - 2(4 - x^2 - y^2)]\, dx\, dy = (4r^2 - 7)r\, dr\, d\theta$.

The flux is $\int_0^{2\pi} \int_0^2 (4r^3 - 7r)\, dr\, d\theta = 2\pi \left[r^4 - \frac{7r^2}{2} \right]_0^2 = 2\pi(16 - 14) = 4\pi$.

The total flux is $4\pi - 4\pi = 0$.

Using divergence theorem. The flux is given by $\iiint_D (\overrightarrow{\nabla} \cdot \overrightarrow{\mathbf{F}}) dV$, where D is the solid region enclosed. In our case $\overrightarrow{\nabla} \cdot \overrightarrow{\mathbf{F}} = 1 + 1 - 2 = 0$, hence the total flux is 0.

Problem 16. $\overrightarrow{\nabla} \times \overrightarrow{\mathbf{F}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \partial_x & \partial_y & \partial_z \\ -6y^2 + 6y & x^2 - 3z^2 & -x^2 \end{vmatrix} = 6z\hat{\boldsymbol{\imath}} + 2x\hat{\boldsymbol{\jmath}} + (2x + 12y - 6)\hat{\boldsymbol{k}}$.

Call R the region of the plane $x + 2y + z = 1$ enclosed by a simple closed curve C lying entirely on that plane. Stokes' theorem: $\int_C \overrightarrow{\mathbf{F}} \cdot d\vec{\boldsymbol{r}} = \iint_R \left(\overrightarrow{\nabla} \times \overrightarrow{\mathbf{F}} \cdot \hat{\mathbf{n}} \right) dS$.

On R we have $\hat{\mathbf{n}} = \frac{\langle 1, 2, 1\rangle}{\sqrt{6}}$ and $\overrightarrow{\nabla} \times \overrightarrow{\mathbf{F}} \cdot \hat{\mathbf{n}} = \frac{6z + 2(2x) + (2x + 12y - 6)}{\sqrt{6}} =$
$= \sqrt{6}(x + 2y + z - 1) = 0$, because R belongs to the plane $x + 2y + z = 1$.

We conclude that $\int_C \overrightarrow{\mathbf{F}} \cdot d\vec{\boldsymbol{r}} = \iint_R \left(\overrightarrow{\nabla} \times \overrightarrow{\mathbf{F}} \cdot \hat{\mathbf{n}} \right) dS = 0$ because $\overrightarrow{\nabla} \times \overrightarrow{\mathbf{F}} \cdot \hat{\mathbf{n}} = 0$.