18.02 Practice Exam 3B

Problem 1. a) Draw a picture of the region of integration of $\int_0^1 \int_x^{2x} dydx$

b) Exchange the order of integration to express the integral in part (a) in terms of integration in the order $dxdy$. Warning: your answer will have two pieces.

Problem 2. a) Find the mass M of the upper half of the annulus, $1 < x^2 + y^2 < 9$ $(y \geq 0)$ with density $\delta = \dfrac{y}{x^2 + y^2}$.

b) Express the x-coordinate of the center of mass, $\bar{x}$, as an iterated integral. (Write explicitly the integrand and limits of integration.) Without evaluating the integral, explain why $\bar{x} = 0$.

Problem 3. a) Show that $\mathbf{F} = (3x^2 - 6y^2)\hat{\boldsymbol{\imath}} + (-12xy + 4y)\hat{\boldsymbol{\jmath}}$ is conservative.

b) Find a potential function for $\mathbf{F}$.

c) Let C be the curve $x = 1 + y^3(1 - y)^3$, $0 \leq y \leq 1$. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Problem 4. a) Express the work done by the force field $\mathbf{F} = (5x + 3y)\hat{\boldsymbol{\imath}} + (1 + \cos y)\hat{\boldsymbol{\jmath}}$ on a particle moving counterclockwise once around the unit circle centered at the origin in the form $\int_a^b f(t)dt$. (**Do not evaluate the integral; don't even simplify** $f(t)$**.**)

b) Evaluate the line integral using Green's theorem.

Problem 5. Consider the rectangle R with vertices $(0, 0)$, $(1, 0)$, $(1, 4)$ and $(0, 4)$. The boundary of R is the curve C, consisting of C_1, the segment from $(0, 0)$ to $(1, 0)$, C_2, the segment from $(1, 0)$ to $(1, 4)$, C_3 the segment from $(1, 4)$ to $(0, 4)$ and C_4 the segment from $(0, 4)$ to $(0, 0)$. Consider the vector field

$$\mathbf{F} = (\cos x \sin y)\hat{\boldsymbol{\imath}} + (xy + \sin x \cos y)\hat{\boldsymbol{\jmath}}$$

a) Find the work of $\mathbf{F}$ along the boundary C oriented in a counterclockwise direction.

b) Is the total work along C_1, C_2 and C_3, more than, less than or equal to the work along C?

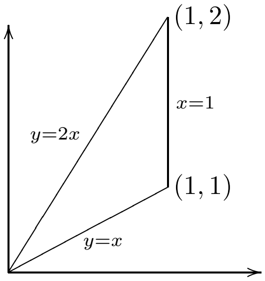
Problem 6. Find the volume of the region enclosed by the plane $z = 4$ and the surface

$$z = (2x - y)^2 + (x + y - 1)^2.$$

Suggestion: change of variables.

18.02 Practice Exam 3B — Solutions

1. a)



b) $\displaystyle\int_0^1 \int_{y/2}^{y} dx dy + \int_1^2 \int_{y/2}^{1} dx dy.$

2. a) $\delta dA = \dfrac{r\sin\theta}{r^2} r dr d\theta = \sin\theta dr d\theta.$

$$M = \int\int_R \delta dA = \int_0^{\pi}\int_1^3 \sin\theta\ dr d\theta = \int_0^{\pi} 2\sin\theta d\theta = \big[-2\cos\theta\big]_0^{\pi} = 4.$$

b) $\displaystyle\bar{x} = \frac{1}{M}\int\int_R x\delta dA = \frac{1}{4}\int_0^{\pi}\int_1^3 r\cos\theta\sin\theta dr d\theta$

The reason why one knows that $\bar{x} = 0$ without computation is that x is odd with respect to the y-axis whereas the region **and the density** are symmetric with respect to the y-axis: $(x, y) \to (-x, y)$ preserves the half annulus and $\delta(x, y) = \delta(-x, y)$.

3. a) $N_x = -12y = M_y$, hence $\mathbf{F}$ is conservative.

b) $f_x = 3x^2 - 6y^2 \Rightarrow f = x^3 - 6y^2x + c(y) \Rightarrow f_y = -12xy + c'(y) = -12xy + 4y$. So $c'(y) = 4y$, thus $c(y) = 2y^2$ (+ constant). In conclusion

$$f = x^3 - 6xy^2 + 2y^2 \quad (+\text{ constant}).$$

c) The curve C starts at $(1, 0)$ and ends at $(1, 1)$, therefore

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(1,1) - f(1,0) = (1 - 6 + 2) - 1 = -4.$$

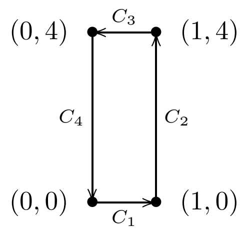
4. a) The parametrization of the circle C is $x = \cos t$, $y = \sin t$, for $0 \le t < 2\pi$; then $dx = -\sin t dt$, $dy = \cos t dt$ and

$$W = \int_0^{2\pi} (5\cos t + 3\sin t)(-\sin t)dt + (1 + \cos(\sin t))\cos t dt.$$

b) Let R be the unit disc inside C;

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int\int_R (N_x - M_y)dA = \int\int_R (0 - 3)dA = -3\ \text{area}(R) = -3\pi.$$

5. a)



$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int\int_R (N_x - M_y)dxdy$$
$$= \int\int_R (y + \cos x \cos y - \cos x \cos y)dxdy = \int\int_R ydxdy$$
$$= \int_0^4 \int_0^1 ydxdy = \int_0^4 ydy = [y^2/2]_0^4 = 8.$$

b) On C_4, $\mathbf{F} = \sin y\ \mathbf{i}$, whereas $d\mathbf{r}$ is parallel to $\mathbf{j}$. Hence $\mathbf{F}\cdot d\mathbf{r} = 0$. Therefore the work of $\mathbf{F}$ along C_4 equals 0. Thus

$$\int_{C_1+C_2+C_3} \mathbf{F}\cdot d\mathbf{r} = \oint_C \mathbf{F}\cdot d\mathbf{r} - \int_{C_4} \mathbf{F}\cdot d\mathbf{r} = \oint_C \mathbf{F}\cdot d\mathbf{r}\ ;$$

and the total work along $C_1 + C_2 + C_3$ is equal to the work along C.

6. Let $u = 2x - y$ and $v = x + y - 1$. The Jacobian $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3.$

Hence $dudv = 3dxdy$ and $dxdy = \frac{1}{3}dudv$, so that

$$\begin{aligned} V &= \int\int_{(2x-y)^2+(x+y-1)^2<4} 4 - (2x-y)^2 - (x+y-1)^2\ dxdy \\ &= \int\int_{u^2+v^2<4} (4 - u^2 - v^2)\frac{1}{3}dudv \\ &= \int_0^{2\pi}\int_0^2 (4 - r^2)\frac{1}{3}rdrd\theta = \int_0^{2\pi} \left[\frac{2}{3}r^2 - \frac{1}{12}r^4\right]_0^2 \\ &= \int_0^{2\pi} \frac{4}{3}d\theta = \frac{8\pi}{3}. \end{aligned}$$