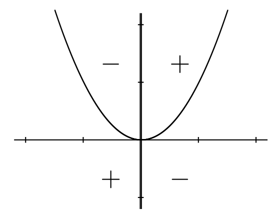
18.02 Practice Exam 2B – Solutions

Problem 1.



a) $f(x,y) = x(y-x^2)$ is zero on the y-axis ($x=0$) and on the parabola $y = x^2$.

b) saddle point.

c) $\nabla f = (y-3x^2)\,\hat{\imath} + x\,\hat{\jmath}$; at P, $\nabla f = \langle -2, 1\rangle$.

d) $\Delta w \simeq -2\,\Delta x + \Delta y$.

Problem 2.

a) By measuring, $\Delta h = 100$ for $\Delta s \simeq 500$, so $\left(\dfrac{dh}{ds}\right)_{\hat{u}} \simeq .2$.

b) Q is the northernmost point on the curve $h = 2200$; $\dfrac{\partial h}{\partial y} \simeq \dfrac{\Delta h}{\Delta y} \simeq \dfrac{-100}{1000/3} \simeq -.3$.

Problem 3.

$f(x,y,z) = x^3y + z^2 = 3$: the normal vector is $\nabla f = \langle 3x^2y, x^3, 2z\rangle = \langle 3,-1,4\rangle$. The tangent plane is $3x - y + 4z = 4$.

Problem 4.

a) The volume is $xyz = xy(1-x^2-y^2) = xy - x^3y - xy^3$. Critical points: $f_x = y - 3x^2y - y^3 = 0$, $f_y = x - x^3 - 3xy^2 = 0$.

b) Assuming $x > 0$ and $y > 0$, the equations can be rewritten as $1-3x^2-y^2 = 0$, $1-x^2-3y^2 = 0$. Solution: $x^2 = y^2 = 1/4$, i.e. $(x,y) = (1/2, 1/2)$.

c) $f_{xx} = -6xy = -3/2$, $f_{yy} = -6xy = -3/2$, $f_{xy} = 1 - 3x^2 - 3y^2 = -1/2$. So $f_{xx}f_{yy} - f_{xy}^2 > 0$, and $f_{xx} < 0$, it is a maximum.

d) $f(x,y,z) = xyz$, $g(x,y,z) = x^2+y^2+z = 1$: must have $\nabla f = \lambda \nabla g$, i.e. $yz = 2\lambda x$, $xz = 2\lambda y$, $xy = \lambda$.

Problem 5.

$\dfrac{\partial w}{\partial x} = f_u u_x + f_v v_x = y\, f_u + \dfrac{1}{y}\, f_v$. $\quad \dfrac{\partial w}{\partial y} = f_u u_y + f_v v_y = x\, f_u - \dfrac{x}{y^2}\, f_v$.

Problem 6.

Chain rule: $\left(\dfrac{\partial w}{\partial z}\right)_y = \dfrac{\partial w}{\partial x}\left(\dfrac{\partial x}{\partial z}\right)_y = 3x^2y\left(\dfrac{\partial x}{\partial z}\right)_y$. To find $\left(\dfrac{\partial x}{\partial z}\right)_y$, differentiate the relation $x^2y + xz^2 = 5$ w.r.t. z holding y constant: $(2xy+z^2)\left(\dfrac{\partial x}{\partial z}\right)_y + 2xz = 0$, so $\left(\dfrac{\partial x}{\partial z}\right)_y = \dfrac{-2xz}{2xy+z^2}$. Therefore $\left(\dfrac{\partial w}{\partial z}\right)_y = \dfrac{-6x^3yz}{2xy+z^2}$. At $(x,y,z) = (1,1,2)$ this is equal to -2.