18.02 Practice Exam 1B Solutions

Problem 1.

a) $P = (1,0,0)$, $Q = (0,2,0)$ and $R = (0,0,3)$. Therefore $\overrightarrow{QP} = \hat{\imath} - 2\hat{\jmath}$ and $\overrightarrow{QR} = \hat{k} - 2\hat{\jmath}$.

b) $\cos\theta = \dfrac{\overrightarrow{QP}\cdot\overrightarrow{QR}}{\left|\overrightarrow{QP}\right|\left|\overrightarrow{QP}\right|} = \dfrac{\langle 1,-2,0\rangle\cdot\langle 0,-2,3\rangle}{\sqrt{1^2+2^2}+\sqrt{2^2+3^2}} = \dfrac{4}{\sqrt{65}}$

Problem 2.

a) $\overrightarrow{PQ} = \langle -1,2,0\rangle$, $\overrightarrow{PR} = \langle -1,0,3\rangle$.

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = 6\hat{\imath} + 3\hat{\jmath} + 2\hat{k}.$$

Then $\quad area(\Delta) = \dfrac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = \dfrac{1}{2}\sqrt{6^2+3^2+2^2} = \dfrac{1}{2}\sqrt{49} = \dfrac{7}{2}.$

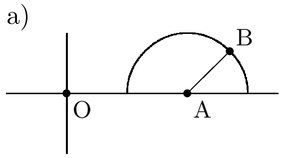
b) A normal to the plane is given by $\overrightarrow{N} = \overrightarrow{PQ}\times\overrightarrow{PR} = \langle 6,3,2\rangle$. Hence the equation has the form $6x+3y+2z = d$. Since P is on the plane $d = 6\cdot 1 + 3\cdot 1 + 2\cdot 1 = 11$. In conclusion the equation of the plane is

$$6x + 3y + 2z = 11.$$

c) The line is parallel to $\langle 2-1, 2-2, 0-3\rangle = \langle 1,0,-3\rangle$. Since $\overrightarrow{N}\cdot\langle 1,0,-3\rangle = 6-6 = 0$, the line is parallel to the plane.

Problem 3.

a)



$\overrightarrow{OA} = \langle 10t, 0\rangle$ and $\overrightarrow{AB} = \langle \cos t, \sin t\rangle$, hence

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \langle 10t + \cos t, \sin t\rangle.$$

The rear bumper is reached at time $t = \pi$ and the position of B is $(10\pi - 1, 0)$.

b) $\overrightarrow{V} = \langle 10 - \sin t, \cos t\rangle$, thus

$$|\overrightarrow{V}|^2 = (10-\sin t)^2 + \cos^2 t = 100 - 20\sin t + \sin^2 t + \cos^2 t = 101 - 20\sin t.$$

The speed is then given by $\sqrt{101 - 20\sin t}$. The speed is smallest when $\sin t$ is largest i.e. $\sin t = 1$. It occurs when $t = \pi/2$. At this time, the position of the bug is $(5\pi, 1)$. The speed is largest when $\sin t$ is smallest; that happens at the times $t = 0$ or π for which the position is then $(0,0)$ and $(10\pi - 1, 0)$.

Problem 4.

a) $|M| = -12$.

b) $a = -5$, $b = 7$.

c) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \dfrac{1}{12}\begin{bmatrix} 1 & 1 & 4 \\ -5 & 7 & -8 \\ 7 & -5 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ t \\ 3 \end{bmatrix} = \begin{bmatrix} t/12 + 1 \\ 7t/12 - 2 \\ -5t/12 + 1 \end{bmatrix}$

d) $\dfrac{d\vec{r}}{dt} = \left\langle \dfrac{1}{12}, \dfrac{7}{12}, -\dfrac{5}{12}\right\rangle.$

Problem 5.

a) $\overrightarrow{N}\cdot\overrightarrow{r}(t) = 6$, where $\overrightarrow{N} = \langle 4,-3,-2\rangle$.

b) We differentiate $\overrightarrow{N}\cdot\overrightarrow{r}(t) = 6$:

$0 = \dfrac{d}{dt}\left(\overrightarrow{N}\cdot\overrightarrow{r}(t)\right) = \dfrac{d}{dt}\overrightarrow{N}\cdot\overrightarrow{r}(t) + \overrightarrow{N}\cdot\dfrac{d}{dt}\overrightarrow{r}(t) = \overrightarrow{0}\cdot\overrightarrow{r}(t) + \overrightarrow{N}\cdot\dfrac{d}{dt}\overrightarrow{r}(t)$ and hence $\overrightarrow{N}\perp\frac{d}{dt}\overrightarrow{r}(t)$.