18.02 Problem Set 8 – Spring 2006

Due Thursday 4/13/06, 12:55 pm

Part A (15 points)

Hand in the underlined problems only; the others are for more practice.

Lecture 22. Thu, Apr 6. Vector fields and line integrals in the plane.
Read: 21.1, Notes V1.
Work: 4A/ 1bd, 2abc, 3abd, 4; 4B/ 1abcd, 2ab. (4B/1c orientation is not the usual one)

Lecture 23. Fri, Apr 7 Gradient fields, path independence, and conservative fields.
Read: 21.2, V2.1
Work: 4C/ 1, 2, 3.

Lecture 24. Tue, Apr 9 Conservative fields continued: finding potential functions.
Read: all of Notes V2
Work: 4C/ 5a (by method 1 of V2), 5b (by method 2), 6ab (by both methods).

Part B (22 points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

Problem 0. (not until due date; 3 points)

Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation".

Problem 1. (Thursday; 5 points: 2 + 3) Let $\boldsymbol{F} = y^2\hat{\boldsymbol{\imath}} + x^2y\hat{\boldsymbol{\jmath}}$

a) Let C_1 be the ellipse $x^2 + 4y^2 = 4$ traced (once) counterclockwise. Compute the line integral $\oint_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r}$. (The circle in the integral is new notation. It is optional: its purpose is to emphasize that the integral is around a loop, that is, a closed curve.)

b) Compute $\oint_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r}$, where C_2 is the triangle in the first quadrant formed by the axes and the line $x + y = a$ traced counterclockwise. ($a > 0$.)

Problem 2. (Friday; 8 points: 1 +2 + 2 + 3)

Consider the vector field $\boldsymbol{F}(x, y) = \dfrac{-y\,\hat{\boldsymbol{\imath}} + x\,\hat{\boldsymbol{\jmath}}}{x^2 + y^2}$.

a) Show that $\boldsymbol{F}$ is the gradient of the polar angle function $\theta(x, y) = \tan^{-1}(y/x)$ defined over the right half-plane $x > 0$.

b) Suppose that C is a smooth curve in the right half-plane $x > 0$ joining two points $A : (x_1, y_1)$ and $B : (x_2, y_2)$. Use the fundamental theorem of calculus (for line integrals) to express $\int_C \boldsymbol{F} \cdot \hat{\boldsymbol{T}}\,ds$ in terms of the polar coordinates (r_1, θ_1) and (r_2, θ_2) of A and B. Find an example of a curve C in the right half-plane for which the line integral is $-\pi/2$.

c) Calculate directly the integrands $\boldsymbol{F} \cdot \hat{\boldsymbol{T}}$ and the integrals $\int_{C_1} \boldsymbol{F} \cdot \hat{\boldsymbol{T}}\,ds$ and $\int_{C_2} \boldsymbol{F} \cdot \hat{\boldsymbol{T}}\,ds$, where C_1 is the upper half of the unit circle running from $(1, 0)$ to $(-1, 0)$, and C_2 is the lower half of the unit circle, also going from $(1, 0)$ to $(-1, 0)$.

d) Since curl $\boldsymbol{F} = 0$ at any point of the plane where $\boldsymbol{F}$ is defined, the vector field $\boldsymbol{F}$ ought to be conservative (path-independent). This is true in some regions, but not in others. Consider three regions, the punctured plane R_1, defined as the whole plane except the origin $(0,0)$, the slit plane R_2, defined as the whole plane except the negative x-axis ($(0,x)$ where $x \leq 0$), and the right half-plane R_3, as defined in part (a). In which of the regions R_1, R_2, and/or R_3 is $\boldsymbol{F}$ conservative? Even though $\boldsymbol{F}$ is not conservative in one of these regions, this does not contradict the fundamental theorem. Explain.

Problem 3. (Tuesday; 3 points: 1 + 2)

a) Calculate the curl of $\boldsymbol{F} = r^n(x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}})$.

b) For each n for which curl $\boldsymbol{F} = 0$, find a potential g such that $\boldsymbol{F} = \nabla g$. (Hint: look for a potential of the form $g = g(r)$. Watch out for a certain negative value of n for which the formula is different.)

Problem 4. (Tuesday; 3 points: 1 + 1 + 1)

a) Verify that $\boldsymbol{F} = y^2\,\hat{\boldsymbol{\imath}} + x^2y\hat{\boldsymbol{\jmath}}$ from Problem 1 is not a gradient field.

b) Try to find a potential function by Method 1 in Notes V2. What goes wrong?

c) Try to find a potential function by Method 2 in Notes V2. What goes wrong?