

## 18.02 Problem Set 6 — Spring 2006

Due Thursday 3/23/06, 12:45 pm

### Part A (15 points)

Hand in the underlined problems only; the others are for more practice.

Lecture 17. Fri, Mar 17. Double integrals and iterated integrals

Read: 20.1, 20.2 and Notes I.1.

Work: 3A/ 1abc, 2abc, 3ab, 4bc, 5ab, 6.

Lecture 18. Tues, Mar 21. Double integrals in polar coordinates. Applications.

Read: 20.3, 20.4, and Notes I.2

Work: 3B/ 1ac, 2a, 3ac; 3C/ 1, 2b, 5; Do, even though not underlined: Find the center of mass of the region in the first quadrant where  $x + y \leq a$  with density  $\delta(x, y) = xy$ .

### Part B (29 points)

**Directions:** Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

**Problem 0.** (not until due date; 3 points) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say “none” or “no consultation”.

**Problem 1.** (Thursday, 13 points: 2 + 1 + 4 + 1 + 2 + 3)

a) Express the line through  $P_0 = (x_0, y_0, z_0)$  in the direction of  $\mathbf{v} = \langle a, b, c \rangle$  in parametric form using the parameter  $\Delta z = z - z_0$ , the change in  $z$ . What condition on  $\mathbf{v}$  is required in order that this parametric representation be possible?

b) Denote  $\mathbf{r} = \langle x, y, z \rangle$ , and  $\Delta \mathbf{r} = \langle \Delta x, \Delta y, \Delta z \rangle = \langle x - x_0, y - y_0, z - z_0 \rangle$ . Find  $\Delta \mathbf{r} / \Delta z$  and  $d\mathbf{r}/dz$  at  $P_0$  along the line in part (a).

c) Consider the functions  $u = x - y + z$  and  $v = -2x + y + 3z$ . Find a parametric representation for the line formed by the intersection of the planes  $u = 1$ ,  $v = 2$ , through the point  $(1, 1, 1)$ , and use the method of part (b) to compute

i)  $\left(\frac{\partial \mathbf{r}}{\partial z}\right)_{u,v}$     ii)  $\left(\frac{\partial \mathbf{r}}{\partial y}\right)_{u,v}$     iii)  $\left(\frac{\partial \mathbf{r}}{\partial x}\right)_{u,v}$

d) What is the relationship between the line in part (c) and lines formed by the intersection of planes  $u = c_1$ ,  $v = c_2$  for other values of  $c_1$  and  $c_2$ ? Use your answer to give a reason why the answers to part (c) would be the same at any base point, not just  $(1, 1, 1)$ .

e) Let  $w = f(x, y, z)$ . Use the chain rule to express, in terms of  $\nabla w = \langle f_x, f_y, f_z \rangle$ ,

i)  $\left(\frac{\partial w}{\partial z}\right)_{u,v}$     ii)  $\left(\frac{\partial w}{\partial y}\right)_{u,v}$     iii)  $\left(\frac{\partial w}{\partial x}\right)_{u,v}$

f) Let  $w_1 = x^3 - zyx$  and  $w_2 = x - y^2z + z^3$ . The surfaces  $w_1 = -7$  and  $w_2 = 49$  intersect in a curve through the point  $P_1 = (1, 2, 4)$ . Find a tangent vector at  $(1, 2, 4)$  to the

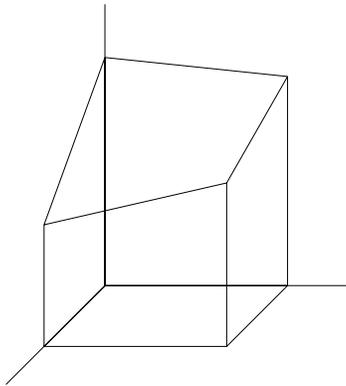
curve, and use it to compute  $dw/dz$  and  $dw/dx$  at  $(x, y, z) = (1, 2, 4)$  in terms of  $\nabla w$ . How are  $dw/dz$  and  $dw/dx$  written in partial derivative notation?

(Compare your answer here for  $dw/dz$  to the one on PS5, Part B/2c. They should be the same if  $w = y$ . In general, partial derivatives are ordinary derivatives along curves in disguise. In order to identify the curve, you need to specify which variables are being held fixed. When computing any rate of change you are, at least implicitly, finding the direction of the curve, as in part (b) and (c) of this problem.)

**Problem 2.** (Friday, 4 points) The rectangular prism pictured below has a rectangular base and a top that is a portion of the plane  $z = ax + by + c$ . The four vertical edges can have different lengths. Show by double integration that

$$\text{volume of prism} = (\text{area of base}) \times (\text{average of the lengths of vertical edges})$$

This formula can be thought of as generalizing the formula for the area of a trapezoid.



**Problem 3.** (Friday, 3 points: 1 + 2) Evaluate  $\int_0^{\infty} \frac{e^{-x} - e^{-ax}}{x} dx$  as follows.

a) Compute  $\int_1^a e^{-xy} dy$ .

b) Use part (a) to rewrite the integral you wish to evaluate as a double integral. Evaluate the double integral by exchanging the order of integration.

**Problem 4.** (Tuesday, 6 points) Compute the mass, centroid, and moments  $I_x$ ,  $I_y$  and  $I_0$  of the half-disk  $y > 0$ ,  $x^2 + y^2 < 1$  with density  $\delta(x, y) = y^2$ . (Use the integral formulas in 3B, if needed.)