

18.02 Problem Set 4

Due Thursday 3/09/06, 12:55 pm

Part A (15 points)

Hand in the underlined problems only; the others are for more practice.

Lecture 10. Thu Mar. 2 Second derivative test. Boundaries and infinity.

Read: 19.7, SD Work: 2H/ 1ac, 3, 4, 6;

Lecture 11. Fri Mar. 3 Differentials. Chain rule.

Read: 19.6 Work: 2C/ 1abcd, 2, 3, 5ab; 2E/ 1abc, 2bc, 5, 8ac.

(Some of the problems in Notes are expressed using notations for gradient: $\nabla f = \text{grad } f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$. See also Section 19.5, the reading for Lecture 12.)

Lecture 12. Tues Mar 7. Gradient, directional derivative, tangent plane, revisited.

Read: 19.5 Work: 2D/ 1abe, 2b, 3ac, 8; 2E/7.

Part B (48 points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

Problem 0. (not until due date; 4 points)

Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say “none” or “no consultation”.

Problem 1. (Thursday, 13 points: 2 + 1 + 4 + 4 + 2) Consider the function $f(x, y) = x^3 - 3x^2 + xy - y^2 + x + y$ in the square S given by $0 \leq x \leq 2$, $0 \leq y \leq 2$.

a) Find the critical points of f in S .

b) Knowing the location of the critical points, what can you say about the points in S where f attains its maximum and minimum?

c) Find the maximum and the minimum of f in S and the points at which these values are achieved.

d) Classify the critical points according to the second derivative test, and sketch the level curves near each such point.

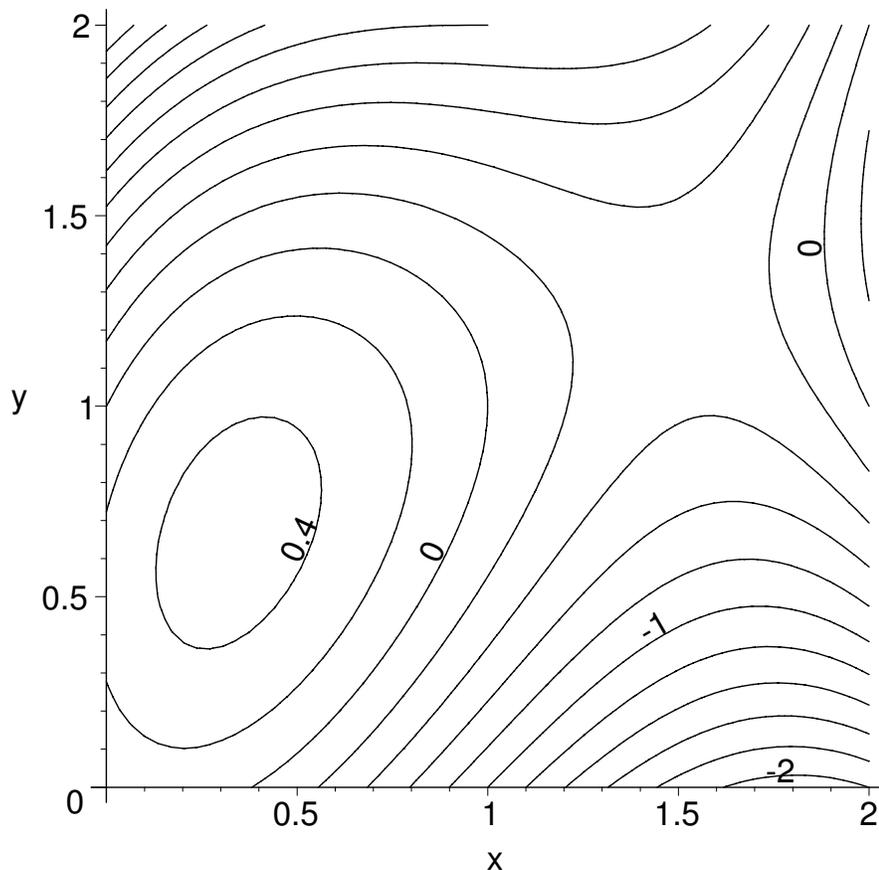
e) What additional information beyond what you found in part (a) does the type of the critical points tell you concerning the location of the maximum and the minimum? If you had carried out the classification of the critical points in part (d) before part (c), what computation(s) would you have been spared? (Hint: something, but not much.)

Problem 2. (Thursday, 12 points: 3 + 1 + 4 + 4)

a) Use the contour plot below to determine whether f_x and f_y are > 0 , $= 0$, or < 0 at the point $(1, \frac{1}{2})$. Same question at the point $(1, 1)$. On which part(s) of the contour line through $(1, 1)$ is $f_x < 0$? (Use the picture — no calculation.)

b) The function plotted on the figure is the function from Problem 1.

$$f(x, y) = x(x - 1)(x - 2) + (y - 1)(x - y) = f(x, y) = x^3 - 3x^2 + xy - y^2 + x + y$$



Calculate the actual values of the partial derivatives at $(1, 1/2)$ and $(1, 1)$.

c) Go to the applet “Functions of two variables” on our web page. The program will load with the function graph of the function f pictured in part (a). Read briefly the directions on the web page, and familiarize yourself with the controls. Choose “Level curves” from the “show” menu, and move to slider to see how the highlighted contour line moves on both the 2-D and 3-D pictures. Experiment with it in order to perfect your understanding of the contour plot. Use the slider to locate (approximately) the critical points and maximum and minimum points. Report all these (approximate) locations and the values of f at these points.

d) Choose “Partial derivatives” from the menu and see how, by clicking on and moving the point in the small contour plot you can see x - and/or y -slices of the graph. [You can use this to check your answers to (b) and to Problem 1.] Find the two points where $f_x = f_y = 0$, and give their approximate coordinates. What happens to the level curves of f through these points? For each of the two points, describe what happens when you move towards North, South, East, West: does the value of f go up, down, or does it stay exactly the same?

Problem 3. (Friday, 3 points: 1 + 2)

The electrical resistance of a piece of wire is given by the formula $R = \rho L/S$, where the resistivity ρ is a constant depending only on the material, L is the length and S is the cross-section area.

a) Show that $dR = R \left(\frac{1}{L} dL - \frac{1}{S} dS \right)$.

b) The electrical resistance of a copper wire of length 100 m and cross-section 1 mm² is equal to 1.5 ohm. Using the differential, approximate the resistance when the length becomes 105 m and the cross-section becomes 1.1 mm². Compare your approximation with the actual value.

Problem 4. (Friday, 8 points 2 + 1 + 2 + 2 + 1; See also, 2E/5,7)

a) Suppose that $w = w(x, y)$, $x = x(r, \theta)$ and $y = y(r, \theta)$. Express the chain rule formula for w_r and w_θ in terms of $w_x, w_y, x_r, y_r, x_\theta, y_\theta$ in matrix notation:

$$\begin{bmatrix} w_r \\ w_\theta \end{bmatrix} = A \begin{bmatrix} w_x \\ w_y \end{bmatrix}$$

(In other words, find the 2×2 matrix A .)

b) Calculate A in the case of polar coordinates $x = r \cos \theta$, $y = r \sin \theta$, writing the entries of A as functions of r and θ .

c) Next suppose that $u = u(r, \theta)$. Use the formulas $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1} y/x$ to derive the chain rule formula

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = B \begin{bmatrix} u_r \\ u_\theta \end{bmatrix}$$

writing the entries of the matrix B as functions of x and y .

d) Compute A^{-1} using the Notes (M, page 6) and change variables to show that $B = A^{-1}$.

e) If at $r = 5$, $\theta = -\pi/2$, $u_r = 1$, $u_\theta = 20$, then what are u_x and u_y ?

Problem 5. (Tuesday, 8 points: 2 + 1 + 1 + 4)

a) Continuing with the same function f as in Problems 1 and 2, Find the maximum and minimum values of the directional derivative $\left. \frac{df}{ds} \right|_{\mathbf{u}}$ at $(1, 1/2)$.

b) Say in which directions \mathbf{u} the maximum and minimum occur.

c) Find the directions \mathbf{u} for which $\left. \frac{df}{ds} \right|_{\mathbf{u}} = 0$.

d) Go to the applet and choose the “Directional derivatives” from the menu. Click to move the point in the contour plot as close as you can to $(1, 1/2)$. ($(1.01, 1.51)$ is ok, but record the actual values of x and y that you use.) Rotate the direction $\mathbf{u} = \langle \cos \theta, \sin \theta \rangle$ and find the numerical values the applet computes for df/ds and θ answering questions (a)-(c) above. In (b) and (c) state the geometric relationship between the yellow direction of the directional derivative \mathbf{u} , the blue contour line ($f = -1/4$), and the purple gradient vector on the contour plot. (The value of the directional derivative is on the lower right in red.)

Problem 6. (Tuesday, 5 points: 2 + 3)

a) Find the direction from $(2, 1, 10)$ in which $g(x, y, z) = z - 2x^2 - y^2$ decreases fastest.

b) Follow the line in the direction you found in part (a) to estimate, using linear approximation, the location of the point closest to $(2, 1, 10)$ at which $g = 0$. Do not use a calculator. Express your answer using fractions. Next, use a calculator to evaluate g at your point. (The value should be reasonably close to 1. Next week, we will get a more precise answer using Lagrange multipliers.)