

18.02 Problem Set 3

Due Thursday 3/02/06, 12:55 pm

Part A (15 points)

Hand in the underlined problems only; the others are for more practice.

Lecture 8. Fri, Feb 24. Functions of several variables, Level curves.

Partial derivatives. Tangent plane approximation.

Read: 19.1–19.3; Notes TA

Work: 2A/ 1abcde, 2abcde, 3abc, 5ab; 2B/ 1ab, 5a, 6, 9.

Confirm that the functions $u(x, t) = \sin(x + at)$ (frequency $a/2\pi$) and $v(x, t) = \cos 7x \sin(7at)$ (frequency $7a/2\pi$) satisfy the equation for the displacement of a vibrating string

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{a^2} \frac{\partial^2 w}{\partial t^2}$$

(This so-called wave equation will be discussed later in Notes P.)

Lecture 9. Tues, Feb 28. Maxima and minima. Least squares.

Read: 19.7; Notes LS

Work: 2F/ 1ab, 2; 2G/ 1ab, 4.

Part B (27 points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

Problem 0. (not until due date; 4 points)

Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say “none” or “no consultation”.

Problem 1. (Friday 3 points) Let f be the height function on the map on the next page, of part of the Presidential range in the White Mountains of New Hampshire. Pretending that the mountains are smooth rather than rough and irregular, estimate the value f_x and f_y at the point on the contour line half way between the “O” in Osgood and the “o” Howker. The scale is 2000 feet per inch and the contour lines are at 100 foot intervals. Use these (very approximate) values of f_x and f_y to estimate the direction of steepest ascent in degrees counterclockwise from north, the vertical direction on the map.

Image of Map removed for copyright reasons.

This map can be found in: [Appalachian Mountain Club White Mountains New Hampshire: Presidential Range](#).

Appalachian Mountain Club, 2003. ISBN: 1929173-26-1.

(By permission, ©Appalachian Mountain Club)

Problem 2. (Thursday, Feb 23, 6 points: 2+1+1+1+1) This problem is designed to introduce you to Matlab, which is a standard computational tool in many courses and engineering and consulting firms.

An economist modeling the ISP wars in the early 21st century has for convenience divided up the companies into four components: AOL, EarthLink, MSN, and Other.com (1stUp.com, NetZero, Prodigy, AT&T Worldnet, Freei.Net, Juno, etc, etc, etc).

The entries x_i of the column vector $\mathbf{x} = [x_1, x_2, x_3, x_4]'$ represent the market share in millions of subscribers of the four “companies” listed above in order; for example $x_2 = 4$ means that Earthlink has 4 million subscribers.

Suppose that after one year, as a result of consumer swiching, 50% of AOL customers have remained loyal, 10% of of Earthlink customers have switched to AOL as have 5% of MSN customers and 15% of others. If y_1, y_2, y_3 and y_4 represent market shares after one year, then we can write

$$y_1 = .5x_1 + .1x_2 + .05x_3 + .15x_4$$

Assuming various switching rates among brands, we get a matrix equation

$$\mathbf{y} = M\mathbf{x}$$

which we will write — changing the names of the column vectors — as

$$\mathbf{x}^{(1)} = M\mathbf{x}^{(0)}$$

(The original market share vector \mathbf{x} is labelled as $\mathbf{x}^{(0)}$, and we change \mathbf{y} to $\mathbf{x}^{(1)}$ to show that it represents the new value of \mathbf{x} after one year.) Let’s say that our economist, on the basis of data

obtained from a telemarketing research firm, assigns to A the value

$$M = \begin{pmatrix} .50 & .10 & .05 & .15 \\ .20 & .60 & .10 & .10 \\ .15 & .10 & .40 & .05 \\ .15 & .20 & .45 & .70 \end{pmatrix}$$

We are supposing for simplicity that the total number subscribers stays the same — a zero-sum game for the ISPs.

Assume that the switching matrix remains the same year after year, so that $\mathbf{x}^{(2)} = M\mathbf{x}^{(1)}$, $\mathbf{x}^{(3)} = M\mathbf{x}^{(2)}$, and so on. The column vector $\mathbf{x}^{(n)}$ summarizes the number of subscribers after n years. The initial market shares in the year 2000 are $x_1 = 22$, $x_2 = 4$, $x_3 = 2.6$, and $x_4 = 25$. Using Matlab (instructions on page 5),

- Calculate $\mathbf{x}^{(n)}$ for $n = 1, 2, 8$ using M^n (round to two decimal places).
- What are the final market shares after many years have gone by? (rounded to two decimal places) What is the first year in which these final market shares appear? (to within 100,000 subscribers)
- Pretend instead that all 53.6 million subscribers started with Earthlink. What will the market shares be after many years?
- Explain briefly why the columns of M all have 1 as their sum.
- How would you modify this model to take into account nonsubscribers? (Be brief; don't complicate matters by worrying about households with more than one service or about population growth.)

Problem 3. (Friday 3 points: 2+1) Assume that $b^2 - 4c > 0$ so that $x^2 + bx + c = 0$ has two roots. Let r denote the larger root. Then r is a function of b and c .

- Give an approximate formula for the small change Δr in the value of r produced by small changes Δb and Δc in the coefficients. Use this to calculate an approximate value for the larger root of the equation $x^2 - 7.01x + 11.98 = 0$.
- Starting from the equation $x^2 - 7x + 12 = 0$, is r more sensitive to small changes in b or c ? (Give a reason.)

Problem 4. (Tuesday, 11 points: 2+0+3+2+3+1) – Least squares and data analysis.

a) Parts (b)-(f) of this problem will involve the use of Matlab. Before going to the terminal, study LS and do the following. Let $\mathbf{x} = [x_1 \ x_2 \ \dots \ x_n]$, $\mathbf{y} = [y_1 \ y_2 \ \dots \ y_n]$ and $\mathbf{u} = [1 \ 1 \ \dots \ 1]$ (n ones). Let $y = ax + b$ be the best-fitting line for the n points (x_i, y_i) . Translate LS (4) into a single 2×2 matrix equation

$$A\mathbf{z} = \mathbf{r}, \quad \mathbf{z} = [a, b]'$$

Write the entries of A and \mathbf{r} in Matlab-ready form. Don't use summations, instead use, for example, $\mathbf{x} * \mathbf{u}'$ for $\sum x_i$. You will be able to confirm that your formulas are correct by testing them on a concrete example using Matlab in part (c).

b) The following data

$$\begin{aligned} x &= [94, 97, 100, 104, 109, 112, 115, 118, 121, 125] \\ y &= [2353, 2601, 2781, 3169, 3547, 3947, 4649, 5050, 5865, 6583] \end{aligned}$$

are taken from World Health Organization statistics about the cumulative number of SARS cases reported (worldwide) from the beginning of the epidemic to a given date, over the period from April 4 to May 5, 2003 (source: <http://www.who.int/csr/sars/country/en/>). The variable x represents the number of days since the beginning of the year, and the variable y represents the total number of cases reported to that date. Look at the Matlab directions for plotting at the end of this problem set and make a scatter plot of these data points marked with *'s. (Nothing to hand in; you will do this over again with part (d).)

c) Use Matlab, the data from (b) and the formulas you found in part (a) to find the best line $y = ax + b$ fitted to the points. Compute the difference between the actual value of the data y with the predicted value $y = ax + b$. Report a , b and the worst case (largest error).

(Optional, but recommended: check that your answer for a and b agrees with the Matlab operation `polyfit(x,y,1)`, which computes the coefficients of the best polynomial of degree 1 fitting the data \mathbf{x} and \mathbf{y} . If $\mathbf{c} = \text{polyfit}(\mathbf{x}, \mathbf{y}, 1)$ then $\mathbf{c} = [a, b] = \mathbf{z}'$ is the transpose of the column vector \mathbf{z} in part (a). In this way you can confirm that you did part (a) correctly.)

d) In its initial phase, an epidemic usually spreads exponentially, not linearly. Use Matlab to find the best fit of the form $\log(y) = a_1x + b_1$. (Use natural log, not log in base 10.) Report your values of a_1 , b_1 . If you exponentiate this equation, you get

$$y = e^{b_1} e^{a_1 x}$$

Compute the difference between y and the predicted value according to this formula and report the worst case (largest error).

e) Hand in a printout that shows on the same plot: the scatter plot of (x, y) labelled with *'s, the straight line fit, $(x, ax + b)$ as a dashed line, and the curve $(x, e^{b_1} e^{a_1 x})$, connected by an ordinary line.

f) According to the exponential best fit, how many cases were there on January 1, 2003? (the order of magnitude is realistic). How many cases would there have been on July 1, 2003 if the epidemic had remained out of control? On December 31, 2003? (fortunately, the epidemic was successfully contained and the total number of cases stabilized around 8400).

Matlab

Matlab calculates with matrices and vectors and draws graphs in 2D and 3D. Skip the Introduction and Help documents; as preliminary practice, just read and carry out the following.

Entering matrices and vectors. In Matlab the variables represent matrices and vectors. The symbol = assigns the value on the right side of the equation to the symbol on the left. Type each of these lines in order, and see what you get. (Always hit [return] to end a line or command.)

```
A = [1 2 3; 4 5 6; 7 8 9]      (you can use commas instead of spaces: 1,2,3;)
b = [5 2 1]
b'      (transpose: gives the column vector which Matlab calls [5;2;1])
eye(3)   (eye = I, the identity matrix)
```

Try making a mistake: `C = [1,2,3; 4,5]`. To edit the mistake, press any of the four arrow keys to get the line back. (You can also prepare your commands in a text editor such as emacs and copy them with the mouse onto the Matlab command line.)

Operations with matrices and vectors

Sum, difference $A+B, A-B$ (matrices must be same size)
Product $A*B$ (matrices must be compatibly sized)
Powers A^n (A times itself n times; A must be square)
Transpose A'
Inverse $\text{inv}(A)$ (or A^{-1})

Try typing (use the values of A and b above): `A+eye(3)` `A*b` `A*(b')` `A*b'` `b*A`

2D graphing with Matlab

Array operations. Recall that `*` and `^` are product and power operations *for matrices*. Adding a dot before `*` or `^` makes these operations act component-wise. So, if $\mathbf{x} = [x_1 \ x_2 \ \dots \ x_n]$, then

$$\begin{aligned}\exp(\mathbf{x}) &= [\exp(x_1) \ \dots \ \exp(x_n)] \quad (\text{similarly with } \sin, \cos, \log, \text{ etc.}) \\ \mathbf{x} + \mathbf{y} &= [x_1 + y_1 \ \dots \ x_n + y_n] \quad (\text{similarly with } -) \\ \mathbf{x} . * \mathbf{y} &= [x_1 y_1 \ \dots \ x_n y_n] \\ \mathbf{x} . ^ m &= [x_1^m \ \dots \ x_n^m] \quad (m \text{ can be zero})\end{aligned}$$

Colon operator. This generates a vector with equally spaced entries; for example,

$$[0 : 2 : 12] = [0 \ 2 \ 4 \ 6 \ 8 \ 10 \ 12]; \quad [2 : -0.1 : 1.6] = [2.0 \ 1.9 \ 1.8 \ 1.7 \ 1.6]$$

2D plot directions. Given $\mathbf{x} = [x_1 \ x_2 \ \dots \ x_n]$, $\mathbf{y} = [y_1 \ \dots \ y_n]$,

`plot(x,y)` plots the n points (x_i, y_i) , joined by solid line segments.

`plot(x,y,'--')` plots the n points, joined by dashed line segments.

`plot(x,y,'*')` plots the n points as individual stars (or dots or circles, etc).

`hold` toggles between on and off (at the start it's off); when off, a new plot erases the previous one; when on, the new plot is superimposed on the old one.

`print` gives a print-out of the current screen plot.

Try in order (press [return] after each command):

```
x=[0:.1:2]
plot(x,sin(x))
plot(x,cos(x),'*')
hold
plot(x,sin(x),'--')
hold
plot(x,4*x.^3)    (this plots  $y = 4x^3$ ; note the need for the array operator)
```

You can also put graphs and scatter plots together without the hold command. The commands below graph the three functions $10x$, $10x^{1/2}$, $2x^{5/3}$. (With the semicolon at the end of each command Matlab won't print out all the numbers. The semicolon also permits you to put several commands on one line.)

```
x = [2:40:400];    w = [1:1:500];    b = 10*(w.^0.5);    c = 2*(w.^(5/3));
plot(x,10*x, '*','w,b,w,c, '--');
```