

18.02 Problem Set 1

Due Thursday 2/16/06, 12:55 pm

Part A (15 points)

Hand in the underlined problems only; the others are for more practice.

(*Notation:* 17.1/7 = Simmons text, Section 17.1, problem 7; 1A/1 = Suppl. Notes E Exercises: 1A-1).

Lecture 1. Tue, Feb 7 Vectors and dot product

Read: 17.3, 18.1, 18.2

Work: 1A/ 1, 5, 6, 7, 8, 9. 1B/ 1, 2, 5ab, 12, 13, 14.

Lecture 2. Thu, Feb 9 Determinants, cross product

Read: Notes D, 18.3

Work: 1C/ 1, 2, 3, 5a, 6, 7.

Lecture 3. Fri, Feb 10 Matrices and inverse matrices

Read: Notes M.1, M.2 (pp. 1–7)

Work: 1D/ 1, 2, 3, 4, 5, 7; 1F/ 5ab, 8a, 9; 1G/ 3, 4, 5.

Lecture 4. Tue, Feb 14 Solving systems of linear equations; lines, planes

Read: Notes M.4 (pp. 9–10), 18.4

Work: 1H/ 3abc, 7; 1E/ 1abcde, 2, 6.

Part B (56 points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

Problem 0. (not until due date; 4 points)

This question will be assigned with every problem set. The purpose of this exercise is to make sure that you acknowledge (to yourself as well as others) what kind of help you require and to encourage you to pay attention to how you learn best (with a tutor, in a group, alone). It will help us by letting us know what resources you use.

Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say “none” or “no consultation”. This includes visits outside recitation to your recitation instructor. If you don’t know a name, you must nevertheless identify the person, as in, “tutor in Room 2-106”, “the student next to me in recitation.” Optional: note which of these people or resources, if any, were particularly helpful to you.

Problem 1. (Tuesday, 5 points: 2+1+2)

The eight vertices of a cube centered at $(0, 0, 0)$ of sidelength 2 are at $(\pm 1, \pm 1, \pm 1)$.

a) Find the four vertices of the cube, starting with $(1, 1, 1)$, that form a regular tetrahedron. Confirm your answer by finding the length of an edge and explaining why all edges have the same length.

b) Find the angle between two adjacent edges (sharing a common vertex) of the tetrahedron.

c) (On Thursday) Find the angle between two adjacent faces of the tetrahedron (sharing a common edge). The angle between two faces is known as a dihedral angle.

Problem 2. (Thursday, 11 points: 3+2+3+3)

a) Find the vector equation for the set of points Q in the plane through the point A with normal \mathbf{N} , and find the distance between parallel planes P and P' with normal \mathbf{N} through A and A' .

b) In part (a), let $\mathbf{N} = \langle 1, 2, 1 \rangle$, $A = (0, -2, 1)$, $A' = (1, 1, 0)$. Write the equation for each of the planes P and P' in the form $ax + by + cz = d$ and find the distance between them.

c) Consider the line L in 3-space through the points B_1 and B_2 and the line L' through the points B'_1 and B'_2 . If the lines are not parallel but do not meet, they are called skew. Assuming the lines are skew, find vector formulas for parallel planes P and P' containing L and L' .

d) Find the planes P and P' in part (c) in the case $B_1 = (1, -1, 0)$, $B_2 = (1, 2, -1)$, $B'_1 = (0, 3, -1)$, $B'_2 = (4, 0, 1)$. Then use the preceding parts to find the (nearest) distance between points on the two lines L and L' .

Problem 3. (Thursday, 10 points: 5+5) *Do you see red?*

The ray from a point Q_0 in the unit direction \mathbf{u} is the set of points Q such that $\overrightarrow{Q_0Q} = t\mathbf{u}$, $t > 0$. (It is also legal to write this equation as $Q = Q_0 + t\mathbf{u}$.) Suppose you are wearing virtual reality headgear at position Q_0 . If you look in the direction \mathbf{u} , then you see the first object you meet along the ray $Q_0 + t\mathbf{u}$, that is, the point with the least $t > 0$. The collection of all unit directions \mathbf{u} forms a sphere. When viewed from Q_0 , a plane P looks like a hemisphere. In other words, the family of directions \mathbf{u} for which the ray $Q_0 + t\mathbf{u}$ meets P is a hemisphere in the sphere of all unit directions \mathbf{u} .

a) Suppose that the universe has two objects in it, namely, the plane P from 2(d) colored red and the plane P' colored blue. What do you see from the point Q_0 ? First describe in words (geometrically) what happens. Split into cases according to where Q_0 is and say whether you see a blue hemisphere, a red one, both, neither, etc. Then describe these hypotheses on Q_0 and conclusions about what \mathbf{u} you see in terms of vector inequalities (not coordinate formulas). For extra credit, discuss what you see in the exceptional cases in which the point Q_0 is on one of the planes.

b) Next consider a universe in which the only two objects are the lines L and L' of 2(d). What do L and L' look like from a given point of view Q_0 ? First describe in words (geometrically) what each line looks like in the sphere of directions \mathbf{u} . Then give a set of rules to decide whether, from the point of view of Q_0 , L and L' cross and, if so, which one is in front. Carry this out explicitly from three viewpoints: $Q_0 = (1, 0, 0)$, $Q_1 = (5, 10, -10)$, and $Q_2 = (1/100, 0, 0)$.

4. (Friday, 8pts; 2+2+2+2) *Rotations*

a) In 2-dimensional space, rotations are given by

$$A_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

Find $\mathbf{u} = A_\theta \hat{\mathbf{i}}$ and $\mathbf{v} = A_\theta \hat{\mathbf{j}}$ and draw a picture of \mathbf{u} and \mathbf{v} for $\theta = \pi/3$.

b) Use the addition formulas for sine and cosine to deduce that $A_{\theta_1}A_{\theta_2} = A_{\theta_1+\theta_2}$. Say in words what this matrix formula means about rotations. (Unlike many matrices, these two-dimensional rotations do commute: $A_{\theta_1}A_{\theta_2} = A_{\theta_2}A_{\theta_1}$.)

c) Calculate A_θ^{-1} by the formula on page 6 of M in the Notes.

d) Give a geometric reason (in terms of the idea of a rotation) why $A_\theta^{-1} = A_{-\theta}$. You found a different-looking formula for A_θ^{-1} in part (c). The fact that the two formulas are equal implies what properties of the sine and cosine functions?

5. (Friday, 10pts; 4 + 6) *Orthogonal matrices*

Recall from 1F/9 that orthogonal matrices are matrices A that satisfy the identity $A(A^T) = I$ (I is the identity matrix). An equivalent definition of the orthogonal matrix property is that $(A^T)A = I$ because the left and right inverses of a square matrix are the same; see 1G/9b.) The

equation $AA^T = I$ says that the rows of A are perpendicular to each other and of unit length, whereas the equation $A^T A = I$ says that the columns of A are perpendicular to each other and of unit length.

The geometric significance of orthogonal matrices is that multiplication by an orthogonal matrix preserves lengths of vectors and the absolute values of angles between them:

$$|A\mathbf{v}| = |\mathbf{v}| \text{ and } |\angle(A\mathbf{v}, A\mathbf{w})| = |\angle(\mathbf{v}, \mathbf{w})|$$

There are two types of orthogonal matrices, rotations and reflections.

a) Find the four orthogonal 2×2 matrices with first entry $a_{11} = 1/\sqrt{2}$. Hint: Try different signs. (See 1F-9 and 1F-10.)

b) Next to each of the four matrices, list the following

i) a picture of what the matrix does to the letter F ,

ii) whether this is a rotation or a reflection,

iii) the determinant of the matrix

What is the pattern?

6. (Tuesday 8pts; 1, 5, 2) Consider the system of equations

$$\begin{array}{rclcl} 2x & - & y & & = & 1 \\ x & + & y & + & 3z & = & 1 \\ x & & & + & 2z & = & 0 \end{array}$$

a) Write the system in matrix form.

b) Find the inverse of the matrix following the method of minors, and solve the equation using this inverse matrix.

c) Solve the system directly using substitution.