18.02 Problem Set 11 Spring 2006

Due Thursday, May 4, 12:55 pm

Part A (10 points)

Hand in the underlined problems only; the others are for more practice.

Lecture 30. Thu Apr 27 Divergence (= Gauss's) theorem.
Read: 21.4, Notes V10
Work: 6C/ 1a, 2, 3, 5, 6, 7a, 8.

Lecture 31. Fri Apr 28 Divergence theorem continued: applications and proof.
Read: Notes V10

Lecture 32. Tue May 2 Line integrals in space, curl, exactness and potentials.
Read: Notes V11, V12
Work: 6D/ 1, 2, 4, 5; 6E/ 1, 2, 3ab(ii) (both methods), 5.

Part B (29 points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

Problem 0. (not until due date; 3 points)

Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation".

Problem 1. (Thursday, 3 points) Notes 6B/7.

Problem 2. (Thursday, 8 points: 2, 4, 2)

Consider a tetrahedron with vertices at $P_0 = (0,0,0)$, $P_1 = (0,1,1)$, $P_2 = (0,1,-1)$, and $P_3 = (1,1,0)$. Orient each face so that the normal vector points out of the tetrahedron.

a) For each of the four faces, determine geometrically (without calculation) whether the flux of the vector field $\mathbf{F} = x\hat{\jmath}$ is positive, negative, or zero.

b) Calculate the flux of $\mathbf{F} = x\hat{\jmath}$ through (out of) the tetrahedron, directly from the definition of flux. (Use symmetry to reduce the amount of calculation.)

c) Verify the answer you found in part (b) by using the divergence theorem.

Problem 3. (Thursday, 6 points) Confirm the divergence theorem in the region $0 \le z \le 10 - \sqrt{x^2+y^2}$ for the vector field $\mathbf{F} = x\hat{\imath} + y\hat{\jmath}$ by calculating each side of the formula directly.

Problem 4. (Thursday, 9 points: 2, 1, 1, 3, 2)

The Laplace operator is $\nabla^2 f = \nabla \cdot \nabla f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$.

A function u is called **harmonic** if $\nabla^2 u = 0$. (The same notations are used in V15 (5)). We will prove the **mean value property** for harmonic functions:

$$u(P) = \frac{1}{4\pi a^2} \iint_{S_a} u \, dS, \quad \text{where } S_a \text{ is the sphere of radius } a \text{ around } P.$$

a) Let S be a surface enclosing a region D. Let $\hat{\boldsymbol{n}}$ to be the unit normal to S pointing outwards from D. The (outer) **normal derivative** of a function g is defined at every point of S as $\dfrac{\partial g}{\partial n} = \hat{\boldsymbol{n}} \cdot \nabla g$. Prove **Green's first identity**,

$$\iint_S f \frac{\partial g}{\partial n}\, dS = \iiint_D (f\nabla^2 g + \nabla f \cdot \nabla g)\, dV$$

by applying the divergence theorem to $\mathbf{F} = f\nabla g$.

b) Use Green's first identity with $f = 1$ and $g = u$, a harmonic function, to deduce that for any closed surface S,

$$\iint_S \frac{\partial u}{\partial n}\, dS = 0.$$

c) Deduce **Green's second identity**,

$$\iint_S (f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n})\, dS = \iiint_D (f\nabla^2 g - g\nabla^2 f)\, dV$$

from Green's first identity by combining Green's first identity with the same formula with f and g interchanged.

d) Let $v = 1/\rho$ and compute $\nabla^2 v$. Let u be any harmonic function. Use Green's second identity in the region $a < \rho < b$ for the functions $f = u$ and $g = v$ to prove that

$$\frac{1}{a^2}\iint_{S_a} u\, dS = \frac{1}{b^2}\iint_{S_b} u\, dS,$$

in which S_r is the sphere of radius r around the origin $\mathbf{0}$.

e) For any continuous function w, $\displaystyle\lim_{a\to 0}\frac{1}{4\pi a^2}\iint_{S_a} w\, dS = w(\mathbf{0})$ (with S_a the sphere centered at the origin). Use this fact and (d) to deduce the mean value property of harmonic functions at $\mathbf{0}$. What region in part (d) and what function in place of v is needed to prove the mean value property at a point P other than the origin?