 The topic for today is going to be equations of planes, and how they relate to linear systems and matrices as we have seen during Tuesday's lecture. So, let's start again with equations of planes. Remember, we've seen briefly that an equation for a plane is of the form ax by cz = d, where a, b, c, and d are just numbers. This expresses the condition for a point at coordinates x, y, z, to be in the plane. An equation of this form defines a plane. Let's see how that works, again. Let's start with an example. Let's say that we want to find the equation of a plane through the origin with normal vector -- -- let's say vector N equals the vector <1,5, 10>. How do we find an equation of this plane? Remember that we can get an equation by thinking geometrically. So, what's our thinking going to be? Well, we have the x, y, z axes. And, we have this vector N: . It's supposed to be perpendicular to our plane. And, our plane passes through the origin here. So, we want to think of the plane that's perpendicular to this vector. Well, when is a point in that plane? Let's say we have a point, P -- -- at coordinates x, y, z. Well, the condition for P to be in the plane should be that we have a right angle here. OK, so P is in the plane whenever OP dot N is 0. And, if we write that explicitly, the vector OP has components x, y, z; N has components 1,5, 10. So that will give us x 5y 10z = 0. That's the equation of our plane. Now, let's think about a slightly different problem. So, let's do another problem. Let's try to find the equation of the plane through the point P0 with coordinates, say, (2,1,-1), with normal vector, again, the same N = <1,5, 10>. How do we find an equation of this thing? Well, we're going to use the same method. In fact, let's think for a second. I said we have our normal vector, N, and it's going to be perpendicular to both planes at the same time. So, in fact, our two planes will be parallel to each other. The difference is, well, before, we had a plane that was perpendicular to N, and passing through the origin. And now, we have a new plane that's going to pass not through the origin but through this point, P0. I don't really know where it is, but let's say, for example, that P0 is here. Then, I will just have to shift my plane so that, instead of passing through the origin, it passes through this new point. How am I going to do that? Well, now, for a point P to be in our new plane, we need the vector no longer OP but P0P to be perpendicular to N. So P is in this new plane if the vector P0P is perpendicular to N. And now, let's think, what's the vector P0P? Well, we take the coordinates of P, and we subtract those of P0. So, that should be x-2, y-1, and z 1, dot product with <1,5, 10> equals 0. Let's expand this. We get (x-2) 5(y-1) 10(z 1) = 0. Let's put the constants on the other side. We get: x 5y 10z equals -- here minus two becomes two, minus five becomes five, ten becomes minus ten. I think we end up with negative three. So, the only thing that changes between these two equations is the constant term on the right-hand side, the thing that I called d. The other common feature is that the coefficients of x, y, and z: one, five, and ten, correspond exactly to the normal vector. That's something you should remember about planes. These coefficients here correspond exactly to a normal vector and, well, this constant term here roughly measures how far you move from... I f you have a plane through the origin, the right-hand side will be zero. And, if you move to a parallel plane, then this number will become something else. Actually, how could we have found that -3 more quickly? Well, we know that the first part of the equation is like this. And we know something else. We know that the point P0 is in the plane. So, if we plug the coordinates of P0 into this, well, x is 2 5 times 1 10 times -1. We get -3. So, in fact, the number we should have here should be minus three so that P0 is a solution. Let me point out -- (I'll put a 1 here again) -- these three numbers: 1,5, 10, are exactly the normal vector. And one way that we can get this number here is by computing the value of the left-hand side at the point P0. We plug in the point P0 into the left hand side. OK, any questions about that? By the way, of course, a plane doesn't have just one equation. It has infinitely many equations because if instead, say, I multiply everything by two, 2x 10y 20z = -6 is also an equation for this plane. That's because we have normal vectors of all sizes -- we can choose how big we make it. Again, the single most important thing here: in the equation ax by cz = d, the coefficients, a, b, c, give us a normal vector to the plane. So, that's why, in fact, what matters to us the most is finding the normal vector. In particular, if you remember, last time I explained something about how we can find a normal vector to a plane if we know points in the plane. Namely, we can take the cross product of two vectors contained in the plane. Let's just do an example to see if we completely understand what's going on. Let's say that I give you the vector with components , and I give you the plane x y 3z = 5. So, do you think that this vector is parallel to the plane, perpendicular to it, neither? I'm starting to see a few votes. OK, I see that most of you are answering number two: this vector is perpendicular to the plane. There are some other answers too. Well, let's try to figure it out. Let's do the example. Say v is <1,2, -1> and the plane is x y 3z = 5. Let's just draw that plane anywhere -- it doesn't really matter. Let's first get a normal vector out of it. Well, to get a normal vector to the plane, what I will do is take the coefficients of x, y, and z. So, that's . So is perpendicular to the plane. How do we get all the other vectors that are perpendicular to the plane? Well, all the perpendicular vectors are parallel to each other. That means that they are just obtained by multiplying this guy by some number. for example, would still be perpendicular to the plane. is also perpendicular to the plane. But now, see, these guys are not proportional to each other. So, V is not perpendicular to the plane. So it's not perpendicular to the plane. Being perpendicular to the plane is the same as being parallel to its normal vector. Now, what about testing if v is, instead, parallel to the plane? Well, it's parallel to the plane if it's perpendicular to N. Let's check. So, let's try to see if v is perpendicular to N. Well, let's do v.N. That's <1,2, - 1> dot <1,1, 3>. You get 1 2 - 3=0. So, yes. If it's perpendicular to N, it means -- It's actually going to be parallel to the plane. OK, any questions? Yes? [QUESTION FROM STUDENT:] When you plug the vector into the plane equation, you get zero. What does that mean? Let's see. If I plug the vector into the plane equation: 1 2-3, well, the left hand side becomes zero. So, it's not a solution of the plane equation. There's two different things here. One is that the point with coordinates (1,2,- 1) is not in the plane. What that tells us is that, if I put my vector V at the origin, then its head is not going to be in the plane. On the other hand, you're right, the left hand side evaluates to zero. What that means is that, if instead I had taken the plane x y 3z = 0, then it would be inside. The plane is x y 3z = 5, so x y 3z = 0 would be a plane parallel to it, but through the origin. So, that would be another way to see that the vector is parallel to the plane. If we move the plane to a parallel plane through the origin, then the endpoint of the vector is in the plane. OK, that's another way to convince ourselves. Any other questions? OK, let's move on. So, last time we learned about matrices and linear systems. So, let's try to think, now, about linear systems in terms of equations of planes and intersections of planes. Remember that a linear system is a bunch of equations -- say, a 3x3 linear system is three different equations. Each of them is the equation of a plane. So, in fact, if we try to solve a system of equations, that means actually we are trying to find a point that is on several planes at the same time. So... Let's say that we have a 3x3 linear system. Just to take an example -- it doesn't really matter what I give you, but let's say I give you x z = 1, x y = 2, x 2y 3z = 3. What does it mean to solve this? It means we want to find x, y, z which satisfy all of these conditions. Let's just look at the first equation, first. Well, the first equation says our point should be on the plane which has this equation. Then, the second equation says that our point should also be on that plane. So, if you just look at the first two equations, you have two planes. And the solutions -- these two equations determine for you two planes, and two planes intersect in a line. Now, what happens with the third equation? That's actually going to be a third plane. So, if we want to solve the first two equations, we have to be on this line. And if we want to solve the third one, we also need to be on another plane. And, in general, the three planes intersect in a point because this line of intersection... Three planes intersect in a point, and one way to think about it is that the line where the first two planes intersect meets the third plane in a point. And, that point is the solution to the linear system. The line -- this is mathematical notation for the intersection between the first two planes -- intersects the third plane in a point, which is going to be the solution. So, how do we find the solution? One way is to draw pictures and try to figure out where the solution is, but that's not how we do it in practice if we are given the equations. Let me use matrix notation. Remember, we saw on Tuesday that the solution to AX = B is given by X = A inverse B. We got from here to there by multiplying on the left by A inverse. A inverse AX simplifies to X equals A inverse B. And, once again, it's A inverse B and not BA inverse. If you try to set up the multiplication, BA inverse doesn't work. The sizes are not compatible, you can't multiply the other way around. OK, that's pretty good -- unless it doesn't work that way. What could go wrong? Well, let's say that our first two planes do intersect nicely in a line, but let's think about the third plane. Maybe the third plane does not intersect that line nicely in a point. Maybe it's actually parallel to that line. Let's try to think about this question for a second. Let's say that the set of solutions to a 3x3 linear system is not just one point. So, we don't have a unique solution that we can get this way. What do you think could happen? OK, I see that answers number three and five seem to be dominating. There's also a bit of answer number one. In fact, these are pretty good answers. I see that some of you figured out that you can answer one and three at the same time, or three and five at the same time. I yet have to see somebody with three hands answer all three numbers at the same time. OK. Indeed, we'll see very soon that we could have either no solution, a line, or a plane. The other answers: "two points" (two solutions), we will see, is actually not a possibility because if you have two different solutions, then the entire line through these two points is also going to be made of solutions. "A tetrahedron" is just there to amuse you, it's actually not a good answer to the question. It's not very likely that you will get a tetrahedron out of intersecting planes. "A plane" is indeed possible, and "I don't know" is still OK for a few more minutes, but we're going to get to the bottom of this, and then we will know. OK, let's try to figure out what can happen. Let me go back to my picture. I had my first two planes; they determine a line. And now I have my third plane. Maybe my third plane is actually parallel to the line but doesn't pass through it. Well, then, there's no solutions because, to solve the system of equations, I need to be in the first two planes. So, that means I need to be in that vertical line. (That line was supposed to be red, but I guess it doesn't really show up as red). And it also needs to be in the third plane. But the line and the plane are parallel to each other. There's just no place where they intersect. So there's no way to solve all the equations. On the other hand, the other thing that could happen is that actually the line is contained in the plane. And then, any point on that line will automatically solve the third equation. So if you try solving a system that looks like this by hand, if you do substitutions, eliminations, and so on, what you will notice is that, after you have dealt with two of the equations, the third one would actually turn out to be the same as what you got out of the first two. It doesn't give you any additional information. It's as if you had only two equations. The previous case would be when actually the third equation contradicts something that you can get out of the first two. For example, maybe out of the first two, you got that x plus z equals one, and the third equation is x plus z equals two. Well, it can't be one and two at the same time. Another way to say it is that this picture is one where you can get out of the equations that a number equals a different number. That's impossible. And, that picture is one where out of the equations you get zero equals zero, which is certainly true, but isn't a very useful equation. So, you can't actually finish solving. OK, let me write that down. unless the third plane is parallel to the line where P1 and P2 intersect. Then there's two subcases. If the line of intersections of P1 and P2 is actually contained in P3 (the third plane), then we have infinitely many solutions. Namely, any point on the line will automatically solve the third equation. The other subcase is if the line of the intersection of P1 and P2 is parallel to P3 and not contained in it. Then we get no solutions. Just to show you the pictures once again: when we have the first two planes, they give us a line. And now, depending on what happens to that line in relation to the third plane, various situations can happen. If the line hits the third plane in a point, then that's going to be our solution. If that line, instead, is parallel to the third plane, well, if it's parallel and outside of it, then we have no solution. If it's parallel and contained in it, then we have infinitely many solutions. So, going back to our list of possibilities, let's see what can happen. No solution: we've seen that it happens when the line where the first two planes intersect is parallel to the third one. Two points: well, that didn't come up. As I said, the problem is that, if the line of intersections of the first two planes has two points that are in the third plane, then that means the entire line must actually be in the third plane. So, if you have two solutions, then you have more than two. In fact, you have infinitely many, and we've seen that can happen. A tetrahedron: still doesn't look very promising. What about a plane? Well, that's a case that I didn't explain because I've been assuming that P1 and P2 are different planes and they intersect in a line. But, in fact, they could be parallel, in which case we already have no solution to the first two equations; or they could be the same plane. And now, if the third plane is also the same plane -- if all three planes are the same plane, then you have a plane of solutions. If I give you three times the same equation, that is a linear system. It's not a very interesting one, but it's a linear system. And "I don't know" is no longer a solution either. OK, any questions? [STUDENT QUESTION:] What's the geometric significance of the plane x y z equals 1, as opposed to 2, or 3? That's a very good question. The question is, what is the geometric significance of an equation like x y z equals to 1,2, 3, or something else? Well, if the equation is x y z equals zero, it means that our plane is passing through the origin. And then, if we change the constant, it means we move to a parallel plane. So, the first guess that you might have is that this number on the right-hand side is the distance between the origin and the plane. It tells us how far from the origin we are. That is not quite true. In fact, that would be true if the coefficients here formed a unit vector. Then this would just be the distance to the origin. Otherwise, you have to actually scale by the length of this normal vector. And, I think there's a problem in the Notes that will show you exactly how this works. You should think of it roughly as how much we have moved the plane away from the origin. That's the meaning of the last term, D, in the right-hand side of the equation. So, let's try to think about what exactly these cases are -- how do we detect in which situation we are? It's all very nice in the picture, but it's difficult to draw planes. In fact, when I draw these pictures, I'm always very careful not to actually pretend to draw an actual plane given by an equation. When I do, then it's blatantly false -- it's difficult to draw a plane correctly. So, instead, let's try to think about it in terms of matrices. In particular, what's wrong with this? Why can't we always say the solution is X = A inverse B? Well, the point is that, actually, you cannot always invert a matrix. Recall we've seen this formula: A inverse is one over determinant of A times the adjoint matrix. And we've learned how to compute this thing: remember, we had to take minors, then flip some signs, and then transpose. That step we can always do. We can always do these calculations. But then, at the end, we have to divide by the determinant. That's fine if the determinant is not zero. But, if the determinant is zero, then certainly we cannot do that. What I didn't mention last time is that the matrix is invertible -- that means it has an inverse -- exactly when its determinant is not zero. That's something we should remember. So, if the determinant is not zero, then we can use our method to find the inverse. And then we can solve using this method. If not, then not. Yes? [STUDENT QUESTION:] Sorry, can you reexplain that? You can invert A if the determinant of A is not equal to zero? That's correct. We can invert the matrix A if the determinant is not zero. If you look again at the method that we saw last time: first we had to compute the adjoint matrix. And, these are operations we can always do. If we are given a 3x3 matrix, we can always compute the adjoint. And then, the last step to find the inverse was to divide by the determinant. And that we can only do if the determinant is not zero. So, if we have a matrix whose determinant is not zero, then we know how to find the inverse. If the determinant is zero, then of course this method doesn't work. I'm actually saying even more: there isn't an inverse at all. It's not just that our method fails. I cannot take the inverse of a matrix with determinant zero. Geometrically, the situation where the determinant is not zero is exactly this nice usual situation where the three planes intersect in a point, while the situation where the determinant is zero is this situation here where the line determined by the first two planes is parallel to the third plane. Let me emphasize this again, and let's see again what happens. Let's start with an easier case. It's called the case of a homogeneous system. It's called homogeneous because it's the situation where the equations are invariant under scaling. So, a homogeneous system is one where the right hand side is zero -- there's no B. If you want, the constant terms here are all zero: 0,0, 0. OK, so this one is not homogenous. So, let's see what happens there. Let's take an example. Instead of this system, we could take x z = 0, x y = 0, and x 2y 3z also equals zero. Can we solve these equations? I think actually you already know a very simple solution to these equations. Yeah, you can just take x, y, and z all to be zero. So, there's always an obvious solution -- -- namely, (0,0, 0). And, in mathematical jargon, this is called the trivial solution. There's always this trivial solution. What's the geometric interpretation? Well, having zeros here means that all three planes pass through the origin. So, certainly the origin is always a solution. The origin is always a solution because the three planes -- -- pass through the origin. Now there's two subcases. One case is if the determinant of the matrix A is nonzero. That means that we can invert A. So, if we can invert A, then we can solve the system by multiplying by A inverse. If we multiply by A inverse, we'll get X equals A inverse times zero, which is zero. That's the only solution because, if AX is zero, then let's multiply by A inverse: we get that A inverse AX, which is X, equals A inverse zero, which is zero. We get that X equals zero. We've solved it, there's no other solution. To go back to these pictures that we all enjoy, it's this case. Now the other case, if the determinant of A equals zero, then this method doesn't quite work. What does it mean that the determinant of A is zero? Remember, the entries in A are the coefficients in the equations. But now, the coefficients in the equations are exactly the normal vectors to the planes. So, that's the same thing as saying that the determinant of the three normal vectors to our three planes is 0. That means that N1, N2, and N3 are actually in a same plane -- they're coplanar. These three vectors are coplanar. So, let's see what happens. I claim it will correspond to this situation here. Let's draw the normal vectors to these three planes. (Well, it's not very easy to see, but I've tried to draw the normal vectors to my planes.) They are all in the direction that's perpendicular to the line of intersection. They are all in the same plane. So, if I try to form a parallelepiped with these three normal vectors, well, I will get something that's completely flat, and has no volume, has volume zero. So the parallelepiped -- -- has volume 0. And the fact that the normal vectors are coplanar tells us that, in fact -- (well, let me start a new blackboard). Let's say that our normal vectors, N1, N2, N3, are all in the same plane. And let's think about the direction that's perpendicular to N1, N2, and N3 at the same time. I claim that it will be the line of intersection. So, let me try to draw that picture again. We have three planes -- (now you see why I prepared a picture in advance. It's easier to draw it beforehand). And I said their normal vectors are all in the same plane. What else do I know? I know that all these planes pass through the origin. So the origin is somewhere in the intersection of the three planes. Now, I said that the normal vectors to my three planes are all actually coplanar. So N1, N2, N3 determine a plane. Now, if I look at the line through the origin that's perpendicular to N1, N2, and N3, so, perpendicular to this red plane here, it's supposed to be in all the planes. (You can see that better on the side screens). And why is that? Well, that's because my line is perpendicular to the normal vectors, so it's parallel to the planes. It's parallel to all the planes. Now, why is it in the planes instead of parallel to them? Well, that's because my line goes through the origin, and the origin is on the planes. So, certainly my line has to be contained in the planes, not parallel to them. So the line through the origin and perpendicular to the plane of N1, N2, N3 -- -- is parallel to all three planes. And, because the planes go through the origin, it's contained in them. So what happens here is I have, in fact, infinitely many solutions. How do I find these solutions? Well, if I want to find something that's perpendicular to N1, N2, and N3 -- if I just want to be perpendicular to N1 and N2, I can take their cross product. So, for example, N1 cross N2 is perpendicular to N1 and to N2, and also to N3, because N3 is in the same plane as N1 and N2, so, if you're perpendicular to N1 and N2, you are also perpendicular to N3. It's automatic. So, it's a nontrivial solution. This vector goes along the line of intersections. OK, that's the case of homogeneous systems. And then, let's finish with the other case, the general case. If we look at a system, AX = B, with B now anything, there's two cases. If the determinant of A is not zero, then there is a unique solution -- -- namely, X equals A inverse B. If the determinant of A is zero, then it means we have the situation with planes that are all parallel to a same line, and then we have either no solution or infinitely many solutions. It cannot be a single solution. Now, whether you have no solutions or infinitely many solutions, we haven't actually developed the tools to answer that. But, if you try solving the system by hand, by elimination, you will see that you end up maybe with something that says zero equals zero, and you have infinitely many solutions. Actually, if you can find one solution, then you know that there's infinitely many. On the other hand, if you end up with something that's a contradiction, like one equals two, then you know there's no solutions. That's the end for today. Tomorrow, we will learn about parametric equations for lines and curves.