 Today I am going to tell you about flux of a vector field for a curve. In case you have seen flux in physics, probably you have seen flux in space, and we are going to come to that in a couple of weeks, but for now we are still doing everything in the plane. So bear with me if you have seen a more complicated version of flux. We are going to do the easy one first. What is flux? Well, flux is actually another kind of line integral. Let's say that I have a plane curve and a vector field in the plane. Then the flux of F across a curve C is, by definition, a line integral, but I will use notation F dot n ds. I have to explain to you what it means, but let me first box that because that is the important formula to remember. That is the definition. What does that mean? First, mostly I have to tell you what this little n is. The notation suggests it is a normal vector, so what does that mean? I have a curve in the plane and I have a vector field. Let's see. The vector field will be yellow today. And I will want to integrate along the curve the dot product of F with the normal vector to the curve, a unit normal vector to the curve. That means a vector that is at every point of the curve perpendicular to the curve and has length one. N everywhere will be the unit normal vector to the curve C pointing 90 degrees clockwise from T. What does that mean? That means I have two normal vectors, one that is pointing this way, one that is pointing that way. I have to choose a convention. And the convention is that the normal vector that I take goes to the right of the curve as I am traveling along the curve. You mentioned that you were walking along this curve, then you look to your right, that is that direction. What we will do is just, at every point along the curve, the dot product between the vector field and the normal vector. And we will sum that along the various pieces of the curve. What this notation means is that if we actually break C into small pieces of length delta(s) then the flux will be the limit, as the pieces become smaller and smaller, of the sum of F dot n delta S. I take each small piece of my curve, I do the dot product between F and n and I multiply by the length of a piece. And then I add these together. That is what the line integral means. Of course that is, again, not how I will compute it. Just to compare this with work, conceptually it is similar to the line integral we did for work except the line integral for work -- Work is the line integral of F dot dr, which is also the line integral of F dot T ds. That is how we reformulated it. That means we take our curve and we figure out at each point how big the tangent component -- I guess I should probably take the same vector field as before. Let's see. My field was pointing more like that way. What I do at any point is project F to the tangent direction, I figure out how much F is going along my curve and then I sum these things together. I am actually summing -- -- the tangential component of my field F. Roughly-speaking the work measures, you know, when I move along my curve, how much I am going with or against F. Flux, on the other hand, measures, when I go along the curve, roughly how much the field is going to across the curve. Counting positively what goes to the right, negatively what goes to the left. Flux is integral F dot n ds, and that one corresponds to summing the normal component of a vector field. But apart from that conceptually it is the same kind of thing. Just the physical interpretations will be very different, but for a mathematician these are two line integrals that you set up and compute in pretty much the same way. Let's see. I should probably tell you what it means. Why do we make this definition? What does it correspond to? Well, the interpretation for work made a lot of sense when F was representing a force. The line integral was actually the work done by the force. The interpretation for flux makes more sense if you think of F as a velocity field. What is the interpretation? Let's say that for F is a velocity field. That means I am thinking of some fluid that is moving, maybe water or something, and it is moving at a certain speed. And my vector field represents how things are moving at every point of the plane. I claim that flux measures how much fluid passes through -- -- the curve C per unit time. If you imagine that maybe you have a river and you are somehow building a damn here, a damn with holes in it so that the water still passes through, then this measures how much water passes through your membrane per unit time. Let's try to figure out why this is true. Why does this make sense? Let's look at what happens on a small portion of our curve C. I am zooming in on my curve C. I guess I need to zoom further. That is a little piece of my curve, of length delta S, and there is a fluid flow. On my picture things are flowing to the right. Here I am drawing a constant vector field because if you zoom in enough then your vectors will pretty much be the same everywhere. If you enlarge the picture enough then things will be pretty much a uniform flow. Now, how much stuff goes through this little piece of curve per unit time? Well, what happens over time is the fluid is moving while my curve is staying the same place so it corresponds to something like this. I claim that what goes through C in unit time is actually going to be a parallelogram. Here is a better picture. I claim that what will be going through C is this shaded parallelogram to the left of C. Let's see. If I move for unit time it works. That is the stuff that goes through my curve, for a small portion of curve in unit time. And, of course, I would need to add all of these together to get the entire curve. Let's try to understand how big this parallelogram is. To know how big this parallelogram is I would like to use base times height or something like that. And maybe I want to actually flip my picture so that the base and the height make more sense to me. Let me actually turn it this way. And, in case you have trouble reading the rotated picture, let me redo it on the board. What passes through a portion of C in unit time is the contents of a parallelogram whose base is on C. So it has length delta s. That is a piece of C. And the other side is going to be given by my velocity vector F. And to find the height of this thing, I need to know what actually the normal component of this vector is. If I call n the unit normal vector to the curve then the area is base times height. The base is delta S and the height is the normal component of F, so it is F dot n. And so you see that when you sum these things together you get, what I said, flux. Now, if you are worried about the fact that actually -- If your unit time is too long then of course things might start changing as it flows. You have to take the time unit and the length unit that are sufficiently small so that really this approximation where C is a straight line and where flow is at constant speed are valid. You want to take maybe a segment here that is a few micrometers. And the time unit might be a few nanoseconds or whatever, and then it is a good approximation. What I mean by per unit time is, well, actually, that works, but you want to think of a really, really small time. And then the amount of matter that passes in that really, really small time is the flux times the amount of time. Let's be a tiny bit more careful. And what I am saying is the amount of stuff that passes through C depends actually on whether n is going this way or the opposite way. Actually, what is implicit in this explanation is that I am counting positively all the stuff that flows across C in the direction of n and negatively what flows in the opposite direction. What flows to the right of C, well, across C from left to right is counted positively. While what flows right to left is counted negatively. So, in fact, it is the net flow through C per unit time. Any questions about the definition or the interpretation or things like that? Yes? Well, you can have both not in the same small segment. But it could be that, well, imagine that my vector field accidentally goes in the opposite direction then this part of the curve, while things are flowing to the left, contributes negatively to flux. And here maybe the field is tangent so the normal component becomes zero. And then it becomes positive and this part of the curve contributes positively. For example, if you imagine that you have a round tank in which the fluid is rotating and you put your dam just on a diameter across then things are going one way on one side, the other way on the other side, and actually it just evens out. We don't have complete information. It is just the total net flux. OK. If there are no other questions then I guess we will need to figure out how to compute this guy and how to actually do this line integral. Well, let's start with a couple of easy examples. Let's say that C is a circle of radius (a) centered at the origin going counterclockwise. And let's say that our vector field is xi yj. What does that look like? Remember, xi plus yj is a vector field that is pointing radially away from the origin. Because at every point it is equal to the vector from the origin to that point. Now, if we have a circle and let's say we are going counterclockwise. Actually, I have a nicer picture. Let me do it here. That is my curve and my vector field. And the normal vector, see, when you go counterclockwise in a closed curve, this convention that a normal vector points to the right of curve makes it point out. The usual convention, when you take flux for a closed curve, is that you are counting the flux going out of the region enclosed by the curve. And, of course, if you went clockwise it would be the other way around. You choose to do it the way you want, but the most common one is to count flux going out of the region. Let's see what happens. Well, if I am anywhere on my circle, see, the normal vector is sticking straight out of the circle. That is a property of the circle that the radial direction is perpendicular to the circle. Actually, let me complete this picture. If I take a point on the circle, I have my normal vector that is pointing straight out so it is parallel to F. Along C we know that F is parallel to n, so F dot n will be equal to the magnitude of F times, well, the magnitude of n, but that is one. Let me put it anywhere, but that is the unit normal vector. Now, what is the magnitude of this vector field if I am at a point x, y? Well, it is square root of x squared plus y squared, which is the same as the distance from the origin. So if this distance, if this radius is a then the magnitude of F will just be a. In fact, F dot n is constant, always equal to a. So the line integral will be pretty easy because all I have to do is the integral of F dot n ds becomes the integral of a ds. (a) is a constant so I can take it out. And integral ds is just a length of C which is 2pi a, so I will get 2pi a squared. And that is positive, as we expected, because stuff is flowing out of the circle. Any questions about that? No. OK. Just out of curiosity, let's say that we had taken our other favorite vector field. Let's say that we had the same C, but now the vector field . Remember, that one goes counterclockwise around the origin. If you remember what we did several times, well, along the circle that vector field now is tangent to the circle. If it is tangent to the circle it doesn't have any normal component. The normal component is zero. Things are not flowing into the circle or out of it. They are just flowing along the circle around and around so the flux will be zero. F now is tangent to C. F dot n is zero and, therefore, the flux will be zero. These are examples where you can compute things geometrically. And I would say, generally speaking, with flux, well, if it is a very complicated field then you cannot. But, if a field is fairly simple, you should be able to get some general feeling for whether your answer should be positive, negative or zero just by thinking about which way is my flow going. Is it going across the curve one way or the other way? Still no questions about these examples? The next thing we need to know is how we will actually compute these things because here, yeah, it works pretty well, but what if you don't have a simple geometric interpretation. What if I give you a really complicated curve and then you have trouble finding the normal vector? It is going to be annoying to set up things this way. Actually, there is a better way to do it in coordinates. Just as we do work, when we compute this line integral, usually we don't do it geometrically like this. Most of the time we just integrate M dx plus N dy in coordinates. That is a similar way to do it because it is, again, a line integral so it should work the same way. Let's try to figure that out. How do we do the calculation in coordinates, or I should say using components? That is the general method of calculation when we don't have something geometric to do. Remember, when we were doing things for work we said this vector dr, or if you prefer T ds, we said just becomes symbolically dx and dy. When you do the line integral of F dot dr you get line integral of n dx plus n dy. Now let's think for a second about how we would express n ds. Well, what is n ds compared to T ds? Well, M is just T rotated by 90 degrees, so n ds is T ds rotated by 90 degrees. That might sound a little bit outrageous because these are really symbolic notations but it works. I am not going to spend too much time trying to convince you carefully. But if you go back to where we wrote this and how we tried to justify this and you work your way through it, you will see that n ds can be analyzed the same way. N is T rotated 90 degrees clockwise. That tells us that n ds is -- How do we rotate a vector by 90 degrees? Well, we swept the two components and we put a minus sign. You have dy and dx. And you have to be careful where to put the minus sign. Well, if you are doing it clockwise, it is in front of dx. Well, actually, let me just convince you quickly. Let's say we have a small piece of C. If we do T delta S, that is also vector delta r. That is going to be just the vector that goes along the curve given by this. Its components will be indeed the change in x, delta x, and the change in y, delta y. And now, if I want to get n delta S, well, I claim now that it is perfectly valid and rigorous to just rotate that by 90 degrees. If I want to rotate this by 90 degrees clockwise then the x component will become the same as the old y component. And the y component will be minus delta x. Then you take the limit when the segment becomes shorter and shorter, and that is how you can justify this. That is the key to computing things in practice. It means, actually, you already know how to compute line integrals for flux. Let me just write it explicitly. Let's say that our vector field has two components. And let me just confuse you a little bit and not call them M and

N for this time just to stress the fact that we are doing a different line integral. Let me call them P and Q for now. Then the line integral of F dot n ds will be the line integral of dot product . That will be the integral of - Q dx P dy. Well, I am running out of space here. It is integral along C of negative Q dx plus P dy. And from that point onwards you just do it the usual way. Remember, here you have two variables x and y but you are integrating along a curve. If you are integrating along a curve x and y are related. They depend on each other or maybe on some other parameter like T or theta or whatever. You express everything in terms of a single variable and then you do a usual single integral. Any questions about that? I see a lot of confused faces so maybe I shouldn't have called my component P and Q. If you prefer, if you are really sentimentally attached to M and N then this new line integral becomes the integral of - N dx M dy. If a problem tells you compute flux instead of saying compute work, the only thing you change is instead of doing M dx plus N dy you do minus N dx plus M dy. And I am sorry to say that I don't have any good way of helping you remember which one of the two gets the minus sign, so you just have to remember this formula by heart. That is the only way I know. Well, you can try to go through this argument again, but it is really best if you just remember that formula. I am not going to do an example because we already know how to do line integrals. Hopefully you will get to see one at least in recitation on Monday. That is all pretty good. Let me tell you now what if I have to compute flux along a closed curve and I don't want to compute it? Well, remember in the case of work we had Green's theorem. We saw yesterday Green's theorem. Let's us replace a line integral along a closed curve by a double integral. Well, here it is the same. We have a line integral along a curve. If it is a closed curve, we should be able to replace it by a double integral. There is a version of Green's theorem for flux. And you will see it is not scarier than the other one. It is perhaps less scarier or perhaps just as scary or just not as scary, depending on how you feel about it, but it works pretty much the same way. What does Green's theorem for flux say? It says if C is a curve that encloses a region R counterclockwise and if I have a vector field that is defined everywhere, not just on C but also inside, so also on R. Well, maybe I should give names to the components. If you will forgive me for a second, I will still use P and Q for now. You will see why. It is defined and differentiable in R. Then I can actually -- -- replace the line integral for flux by a double integral over R of some function. And that function is called the divergence of F dA. This is the divergence of F. And I have to define for you what this guy is. The divergence of a vector field with components P and Q is just P sub x Q sub y. This one is actually easier to remember than curl because you just take the x component, take its partial with respect to x, take the y component, take its partial with respect to y and add them together. No signs. No switching things around. This one is pretty straightforward. The picture again is if I have my curve C going counterclockwise around a region R and I want to find the flux of some vector field F that is everywhere in here. Maybe some parts of C will contribute positively and some parts will contribute negatively. Just to reiterate what I said, positively here means, because we are going counterclockwise, the normal vector points out of the region. This guy here is the flux out of R through C. That is the formula. Any questions about what the statement says or how to use it concretely? No. OK. It is pretty similar to Green's theorem for work. Actually, I should say -- This is called Green's theorem in normal form also. Not that the other one is abnormal, but just that the old one for work was, you could say, in tangential form. That just means, well, Green's theorem, as seen yesterday was for the line integral F dot T ds, integrating the tangent component of F. The one today is for integrating the normal component of F. OK. Let's prove this. Good news. It is much easier to prove than the one we did yesterday because we are just going to show that it is the same thing just using different notations. How do I prove it? Well, maybe actually it would help if first, before proving it, I actually rewrite what it means in components. We said the line integral of F dot n ds is actually the line integral of - Q dx P dy. And we want to show that this is equal to the double integral of P sub x Q sub y dA. This is really one of the features of Green's theorem. No matter which form it is, it relates a line integral to a double integral. Let's just try to see if we can reduce it to the one we had yesterday. Let me forget what these things mean physically and just focus on the math. On the math it is a line integral of something dx plus something dy. Let's call this guy M and let's call this guy N. Let M equal negative Q and N equal P. Then this guy here becomes integral of M dx plus N dy. And I know from yesterday what this is equal to, namely using the tangential form of Green's theorem. Green for work. This is the double integral of curl of this guy. That is Nx minus My dA. But now let's think about what this is in terms of M and N. Well, we said that M is negative Q so this is negative My. And we said P is the same as N, so this is Nx. Just by renaming the components, I go from one form to the other one. So it is really the same theorem. That's why it is also called Green's theorem. But the way we think about it when we use it is different, because one of them computes the work done by a force along a closed curve, the other one computes the flux maybe of a velocity field out of region. Questions? Yes? That is correct. If you are trying to compute a line integral for flux, wait, where did I put it? A line integral for flux just becomes this. And once you are here you know how to compute that kind of thing. The double integral side does not even have any kind of renaming to do. You know how to compute a double integral of a function. This is just a particular kind of function that you get out of a vector field, but it is like any function. The way you would evaluate these double integrals is just the usual way. Namely, you have a function of x and y, you have a region and you set up the bounds for the isolated integral. The way you would evaluate the double integrals is really the usual way, by slicing the region and setting up the bounds for iterated integrals in dx, dy or dydx or maybe rd, rd theta or whatever you want. In fact, in terms of computing integrals, we just have two sets of skills. One is setting up and evaluating double integrals. The other one is setting up and evaluating line integrals. And whether these line integrals or double integrals are representing work, flux, integral of a curve, whatever, the way that we actually compute them is the same. Let's do an example. Oh, first. Sorry. This renaming here, see, that is why actually I call my components P and Q because the argument would have gotten very messy if I had told you now I call M ,N and I call N minus M and so on. But, now that we are through with this, if you still like M and N better, you know, what this says -- The formulation of Green's theorem in this language is just integral of minus N dx plus M dy is the double integral over R of Mx plus Ny dA. Now let's do an example. Let's look at this picture again, the flux of xi plus yj out of the circle of radius A. We did the calculation directly using geometry, and it wasn't all that bad. But let's see what Green's theorem does for us here. Example. Let's take the same example as last time. F equals xi yj. C equals circle of radius a counterclockwise. How do we set up Green's theorem. Well, let's first figure out the divergence of F. The divergence of this field, I take the x component, which is x, and I take its partial respect to x. And then I do the same with the y component, and I will get one plus one equals two. So, the divergence of this field is two. Now, Green's theorem tells us that the flux out of this region is going to be the double integral of 2 dA. What is R now? Well, R is the region enclosed by C. So if C is the circle, R is the disk of radius A. Of course, we can compute it, but we don't have to because double integral of 2dA is just twice the double integral of dA so it is twice the area of R. And we know the area of a circle of radius A. That is piA2. So, it is 2piA2. That is the same answer that we got directly, which is good news. Now we can even do better. Let's say that my circle is not at the origin. Let's say that it is out here. Well, then it becomes harder to calculate the flux directly. And it is harder even to guess exactly what will happen because on this side here the vector field will go into the region so the contribution to flux will be negative here. Here it will be positive because it is going out of the region. There are positive and negative terms. Well, it looks like positive should win because here the vector field is much larger than over there. But, short of computing it, we won't actually know what it is. If you want to do it by direct calculation then you have to parametize this circle and figure out what the line integral will be. But if you use Green's theorem, well, we never used the fact that it is the circle of radius A at the origin. It is true actually for any closed curve that the flux out of it is going to be twice the area of the region inside. It still will be 2piA2 even if my circle is anywhere else in the plane. If I had asked you a trick question where do you want to place this circle so that that the flux is the largest? Well, the answer is it doesn't matter. Now, let's just finish quickly by answering a question that some of you, I am sure, must have, which is what does divergence mean and what does it measure? I mean, we said for curl, curl measures how much things are rotating somehow. What does divergence mean? Well, the answer is divergence measures how much things are diverging. Let's be more explicit. Interpretation of divergence. You can think of it, you know, what do I want to say first? If you take a vector field that is a constant vector field where everything just translates then there is no divergence involved because the derivatives will be zero. If you take the guy that rotates things around you will also compute and find zero for divergence. This is not sensitive to translation motions where everything moves together or to rotation motions, but instead it is sensitive to explaining motions. A possible answer is that it measures how much the flow is expanding areas. If you imagine this flow that we have here on the picture, things are moving away from the origin and they fill out the plane. If we mention this fluid flowing out there, it is occupying more and more space. And so that is what it means to have positive divergence. If you took the opposite vector field that contracts everything to the origin that will have negative divergence. That is a good way to think about it if you are thinking of a gas maybe that can expand to fill out more volume. If you thinking of water, well, water doesn't really shrink or expand. The fact that it is taking more and more space actually means that there is more and more water. The other way to think about it is divergence is the source rate, it is the amount of fluid that is being inserted into the system, that is being pumped into the system per unit time per unit area. What div F equals two here means is that here you actually have matter being created or being pumped into the system so that you have more and more water filling more and more space as it flows. But, actually, divergence is not two just at the origin. It is two everywhere. So, in fact, to have this you need to have a system of pumps that actually is in something water absolutely everywhere uniformly. That is the only way to do this. I mean if you imagine that you just have one spring at the origin then, sure, water will flow out, but as you go further and further away it will do so more and more slowly. Well, here it is flowing away faster and faster. And that means everywhere you are still pumping more water into it. So, that is what divergence measures.