18.02 Multivariable Calculus
Fall 2007

**18.02 Lecture** 14. – **Thu, Oct 11, 2007**

Handouts: PS5 solutions, PS6, practice exams 2A and 2B.

**Non-independent variables.**

Often we have to deal with non-independent variables, e.g. $f(P, V, T)$ where $PV = nRT$.

Question: if $g(x, y, z) = c$ then can think of $z = z(x, y)$. What are $\partial z/\partial x$, $\partial z/\partial y$?

Example: $x^2 + yz + z^3 = 8$ at $(2, 3, 1)$. Take differential: $2x\,dx + z\,dy + (y + 3z^2)\,dz = 0$, i.e. $4\,dx + dy + 6\,dz = 0$ (constraint $g = c$), or $dz = -\frac{4}{6}\,dx - \frac{1}{6}\,dy$. So $\partial z/\partial x = -4/6 = -2/3$ and $\partial z/\partial y = -1/6$ (taking the coefficients of $dx$ and $dy$). Or equivalently: if $y$ is held constant then we substitute $dy = 0$ to get $dz = -4/6\,dx$, so $\partial z/\partial x = -4/6 = -2/3$.

In general: $g(x, y, z) = c \Rightarrow g_x\,dx + g_y\,dy + g_z\,dz = 0$. If $y$ held fixed, get $g_x\,dx + g_z\,dz = 0$, i.e. $dz = -g_x/g_z\,dx$, and $\partial z/\partial x = -g_x/g_z$.

Warning: notation can be dangerous! For example:

$f(x, y) = x + y$, $\partial f/\partial x = 1$. Change of variables $x = u$, $y = u + v$ then $f = 2u + v$, $\partial f/\partial u = 2$.

$x = u$ but $\partial f/\partial x \neq \partial f/\partial u$ !!

This is because $\partial f/\partial x$ means change $x$ keeping $y$ fixed, while $\partial f/\partial u$ means change $u$ keeping $v$ fixed, i.e. change $x$ keeping $y - x$ fixed.

When there's ambiguity, we must precise what is held fixed: $\left(\frac{\partial f}{\partial x}\right)_y$ = deriv. / $x$ with $y$ held fixed, $\left(\frac{\partial f}{\partial u}\right)_v$ = deriv. / $u$ with $v$ held fixed.

We now have $\left(\frac{\partial f}{\partial u}\right)_v = \left(\frac{\partial f}{\partial x}\right)_v \neq \left(\frac{\partial f}{\partial x}\right)_y$.

In above example, we computed $(\partial z/\partial x)_y$. When there is no risk of confusion we keep the old notation, by default $\partial/\partial x$ means we keep $y$ fixed.

**Example:** area of a triangle with 2 sides $a$ and $b$ making an angle $\theta$ is $A = \frac{1}{2}ab\sin\theta$. Suppose it's a right triangle with $b$ the hypothenuse, then constraint $a = b\cos\theta$.

3 ways in which rate of change of $A$ w.r.t. $\theta$ makes sense:

1) view $A = A(a, b, \theta)$ independent variables, usual $\frac{\partial A}{\partial \theta} = A_\theta$ (with $a$ and $b$ held fixed). This answers the question: $a$ and $b$ fixed, $\theta$ changes, triangle stops being a right triangle, what happens to $A$?

2) constraint $a = b\cos\theta$, keep $a$ fixed, change $\theta$, while $b$ does what it must to satisfy the constraint: $\left(\frac{\partial A}{\partial \theta}\right)_a$.

3) constraint $a = b\cos\theta$, keep $b$ fixed, change $\theta$, while $a$ does what it must to satisfy the constraint: $\left(\frac{\partial A}{\partial \theta}\right)_b$.

How to compute e.g. $(\partial A/\partial\theta)_a$? [treat $A$ as function of $a$ and $\theta$, while $b = b(a, \theta)$.]

0) Substitution: $a = b\cos\theta$ so $b = a\sec\theta$, $A = \frac{1}{2}ab\sin\theta = \frac{1}{2}a^2\tan\theta$, $(\frac{\partial A}{\partial\theta})_a = \frac{1}{2}a^2\sec^2\theta$. (Easiest here, but it's not always possible to solve for $b$)

1) Total differentials: $da = 0$ ($a$ fixed), $dA = A_\theta d\theta + A_a da + A_b db = \frac{1}{2}ab\cos\theta\,d\theta + \frac{1}{2}b\sin\theta\,da + \frac{1}{2}a\sin\theta\,db$, and constraint $\Rightarrow da = \cos\theta\,db - b\sin\theta\,d\theta$. Plugging in $da = 0$, we get $db = b\tan\theta\,d\theta$

and then

$$dA = (\frac{1}{2}\,ab\cos\theta + \frac{1}{2}a\sin\theta\, b\tan\theta)d\theta, \quad \left(\frac{\partial A}{\partial\theta}\right)_a = \frac{1}{2}ab\cos\theta + \frac{1}{2}a\sin\theta\, b\tan\theta = \frac{1}{2}ab\sec\theta.$$

2) Chain rule: $(\partial A/\partial\theta)_a = A_\theta(\partial\theta/\partial\theta)_a + A_a(\partial a/\partial\theta)_a + A_b(\partial b/\partial\theta)_b = A_\theta + A_b(\partial b/\partial\theta)_a$. We find $(\partial b/\partial\theta)_a$ by using the constraint equation. [Ran out of time here]. Implicit differentiation of constraint $a = b\cos\theta$: we have $0 = (\partial a/\partial\theta)_a = (\partial b/\partial\theta)_a\cos\theta - b\sin\theta$, so $(\partial b/\partial\theta)_a = b\tan\theta$, and hence

$$\left(\frac{\partial A}{\partial\theta}\right)_a = \frac{1}{2}ab\cos\theta + \frac{1}{2}a\sin\theta\, b\tan\theta = \frac{1}{2}ab\sec\theta.$$

The two systematic methods essentially involve calculating the same quantities, even though things are written differently.

**18.02 Lecture** 15. – **Fri, Oct 12, 2007**

**Review topics.**

– Functions of several variables, contour plots.

– Partial derivatives, gradient; approximation formulas, tangent planes, directional derivatives.

Note: *partial differential equations* (= equations involving partial derivatives of an unknown function) are very important in physics. E.g., heat equation: $\partial f/\partial t = k(\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2)$ describes evolution of temperature over time.

– Min/max problems: critical points, 2nd derivative test, checking boundary.
(least squares won't be on the exam)

– Differentials, chain rule, change of variables.

– Non-independent variables: Lagrange multipliers, and constrained partial derivatives.

Re-explanation of how to compute constrained partials: say $f = f(x, y, z)$ where $g(x, y, z) = c$. To find $(\partial f/\partial z)_y$:

1) using differentials: $df = f_x\,dx + f_y\,dy + f_z\,dz$. We set $dy = 0$ since $y$ held constant, and want to eliminate $dx$. For this we use the constraint: $dg = g_x\,dx + g_y\,dy + g_z\,dz = 0$, so setting $dy = 0$ we get $dx = -g_z/g_x\,dz$. Plug into $df$: $df = -f_x g_z/g_x\,dz + g_z\,dz$, so $(\partial f/\partial z)_y = -f_x g_z/g_x + g_z$.

2) using chain rule: $\left(\frac{\partial f}{\partial z}\right)_y = \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial z}\right)_y + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial z}\right)_y + \frac{\partial f}{\partial z}\left(\frac{\partial z}{\partial z}\right)_y = f_x\left(\frac{\partial x}{\partial z}\right)_y + f_z$, while

$$0 = \left(\frac{\partial g}{\partial z}\right)_y = \frac{\partial g}{\partial x}\left(\frac{\partial x}{\partial z}\right)_y + \frac{\partial g}{\partial y}\left(\frac{\partial y}{\partial z}\right)_y + \frac{\partial g}{\partial z}\left(\frac{\partial z}{\partial z}\right)_y = g_x\left(\frac{\partial x}{\partial z}\right)_y + g_z$$

which gives $(\partial x/\partial z)_y$ and hence the answer.