18.02 Multivariable Calculus
Fall 2007

**18.02 Lecture 8. – Tue, Sept 25, 2007**

**Functions of several variables.**

Recall: for a function of 1 variable, we can plot its graph, and the derivative is the slope of the tangent line to the graph.

Plotting graphs of functions of 2 variables: examples $z = -y$, $z = 1 - x^2 - y^2$, using slices by the coordinate planes. (derived carefully).

Contour plot: level curves $f(x, y) = c$. Amounts to slicing the graph by horizontal planes $z = c$.

Showed 2 examples from "real life": a topographical map, and a temperature map, then did the examples $z = -y$ and $z = 1 - x^2 - y^2$. Showed more examples of computer plots ($z = x^2 + y^2$, $z = y^2 - x^2$, and another one).

Contour plot gives some qualitative info about how $f$ varies when we change $x, y$. (shown an example where increasing $x$ leads $f$ to increase).

**Partial derivatives.**

$f_x = \dfrac{\partial f}{\partial x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$; same for $f_y$.

Geometric interpretation: $f_x, f_y$ are slopes of tangent lines of vertical slices of the graph of $f$ (fixing $y = y_0$; fixing $x = x_0$).

How to compute: treat $x$ as variable, $y$ as constant.

Example: $f(x, y) = x^3 y + y^2$, then $f_x = 3x^2 y$, $f_y = x^3 + 2y$.

**18.02 Lecture 9. – Thu, Sept 27, 2007**

Handouts: PS3 solutions, PS4.

**Linear approximation**

Interpretation of $f_x$, $f_y$ as slopes of *slices* of the graph by planes parallel to $xz$ and $yz$ planes.

Linear approximation formula: $\Delta f \approx f_x \Delta x + f_y \Delta y$.

Justification: $f_x$ and $f_y$ give slopes of two lines tangent to the graph:

$y = y_0,\ z = z_0 + f_x(x_0, y_0)(x - x_0)$ and $x = x_0,\ z = z_0 + f_y(x_0, y_0)(y - y_0)$.

We can use this to get the equation of the tangent plane to the graph:

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Approximation formula = the graph is close to its tangent plane.

**Min/max problems.**

At a local max or min, $f_x = 0$ and $f_y = 0$ (since $(x_0, y_0)$ is a local max or min of the slice). Because 2 lines determine tangent plane, this is enough to ensure that tangent plane is horizontal (approximation formula: $\Delta f \simeq 0$, or rather, $|\Delta f| \ll |\Delta x|, |\Delta y|$).

Def of critical point: $(x_0, y_0)$ where $f_x = 0$ and $f_y = 0$.

A critical point may be a local min, local max, or saddle.

Example: $f(x, y) = x^2 - 2xy + 3y^2 + 2x - 2y$.

Critical point: $f_x = 2x - 2y + 2 = 0$, $f_y = -2x + 6y - 2 = 0$, gives $(x_0, y_0) = (-1, 0)$ (only one critical point).

Is it a max, min or saddle? (pictures shown of each type). Systematic answer: next lecture.

For today: observe $f = (x-y)^2 + 2y^2 + 2x - 2y = (x-y+1)^2 + 2y^2 - 1 \geq -1$, so minimum.

**Least squares.**

Set up problem: experimental data $(x_i, y_i)$ $(i = 1, \ldots, n)$, want to find a best-fit line $y = ax + b$ (the unknowns here are $a, b$, not $x, y$!)

Deviations: $y_i - (ax_i + b)$; want to minimize the total square deviation $D = \sum_i (y_i - (ax_i + b))^2$.

$\dfrac{\partial D}{\partial a} = 0$ and $\dfrac{\partial D}{\partial b} = 0$ leads to a $2 \times 2$ linear system for $a$ and $b$ (done in detail as in Notes LS):

$$\left(\sum x_i^2\right) a + \left(\sum x_i\right) b = \sum x_i y_i$$
$$\left(\sum x_i\right) a + nb = \sum y_i$$

Least-squares setup also works in other cases: e.g. exponential laws $y = ce^{ax}$ (taking logarithms: $\ln y = \ln c + ax$, so setting $b = \ln c$ we reduce to linear case); or quadratic laws $y = ax^2 + bx + c$ (minimizing total square deviation leads to a $3 \times 3$ linear system for $a, b, c$).

Example: Moore's Law (number of transistors on a computer chip increases exponentially with time): showed interpolation line on a log plot.

**18.02 Lecture** 10. – **Fri, Sept 28, 2007**

**Second derivative test.**

Recall critical points can be local min ($w = x^2 + y^2$), local max ($w = -x^2 - y^2$), saddle ($w = y^2 - x^2$); slides shown of each type.

Goal: determine type of a critical point, and find the global min/max.

Note: global min/max may be either at a critical point, or on the boundary of the domain/at infinity.

We start with the case of $w = ax^2 + bxy + cy^2$, at $(0, 0)$.

Example from Tuesday: $w = x^2 - 2xy + 3y^2$: completing the square, $w = (x-y)^2 + 2y^2$, minimum.

If $a \neq 0$, then $w = a(x^2 + \frac{b}{a}xy) + cy^2 = a(x + \frac{b}{2a}y)^2 + (c - \frac{b^2}{4a})y^2 = \frac{1}{4a}(4a^2(x + \frac{b}{2a}y)^2 + (4ac - b^2)y^2)$.

3 cases: if $4ac - b^2 > 0$, same signs, if $a > 0$ then minimum, if $a < 0$ then maximum; if $4ac - b^2 < 0$, opposite signs, saddle; if $4ac - b^2 = 0$, degenerate case.

This is related to the quadratic formula: $w = y^2\big(a(\frac{x}{y})^2 + b(\frac{x}{y}) + c\big)$.

If $b^2 - 4ac < 0$ then no roots, so $at^2 + bt + c$ has a constant sign, and $w$ is either always nonnegative or always nonpositive (min or max). If $b^2 - 4ac > 0$ then $at^2 + bt + c$ crosses zero and changes sign, so $w$ can have both signs, saddle.

**General case:** second derivative test.

We look at second derivatives: $f_{xx} = \dfrac{\partial^2 f}{\partial x^2}$, $f_{xy}$, $f_{yx}$, $f_{yy}$. Fact: $f_{xy} = f_{yx}$.

Given $f$ and a critical point $(x_0, y_0)$, set $A = f_{xx}(x_0, y_0)$, $B = f_{xy}(x_0, y_0)$, $C = f_{yy}(x_0, y_0)$, then:

– if $AC - B^2 > 0$ then: if $A > 0$ (or $C$), local min; if $A < 0$, local max.

– if $AC - B^2 < 0$ then saddle.

– if $AC - B^2 = 0$ then can't conclude.

Checked quadratic case ($f_{xx} = 2a = A$, $f_{xy} = b = B$, $f_{yy} = 2c = C$, then $AC - B^2 = 4ac - b^2$).

General justification: quadratic approximation formula (Taylor series at order 2):

$\Delta f \simeq f_x\,(x - x_0) + f_y\,(y - y_0) + \frac{1}{2} f_{xx}\,(x - x_0)^2 + f_{xy}\,(x - x_0)(y - y_0) + \frac{1}{2} f_{yy}\,(y - y_0)^2$.

At a critical point, $\Delta f \simeq \frac{A}{2}(x - x_0)^2 + B(x - x_0)(y - y_0) + \frac{C}{2}(y - y_0)^2$. In degenerate case, would need higher order derivatives to conclude.

NOTE: the global min/max of a function is not necessarily at a critical point! Need to check boundary / infinity.

Example: $f(x, y) = x + y + \frac{1}{xy}$, for $x > 0$, $y > 0$.

$f_x = 1 - \frac{1}{x^2 y} = 0$, $f_y = 1 - \frac{1}{xy^2} = 0$. So $x^2 y = 1$, $xy^2 = 1$, only critical point is $(1, 1)$.

$f_{xx} = 2/x^3 y$, $f_{xy} = 1/x^2 y^2$, $f_{yy} = 2/xy^3$. So $A = 2$, $B = 1$, $C = 2$.

Question: type of critical point? Answer: $AC - B^2 = 2 \cdot 2 - 1 > 0$, $A = 2 > 0$, local min.

What about the maximum? Answer: $f \to \infty$ near boundary ($x \to 0$ or $y \to 0$) and at infinity.