18.02 Multivariable Calculus
Fall 2007

18.02 Lecture 30. – **Tue, Nov 27, 2007**

Handouts: practice exams 4A and 4B.

Clarification from end of last lecture: we derived the diffusion equation from 2 inputs. u = concentration, $\mathbf{F}$ = flow, satisfy:

1) from physics: $\mathbf{F} = -k\nabla u$,

2) from divergence theorem: $\partial u/\partial t = -\text{div}\,\mathbf{F}$.

Combining, we get the diffusion equation: $\partial u/\partial t = -\text{div}\,\mathbf{F} = +k\text{div}\,(\nabla u) = k\nabla^2 u$.

Line integrals in space.

Force field $\mathbf{F} = \langle P, Q, R\rangle$, curve C in space, $d\vec{r} = \langle dx, dy, dz\rangle$

$\Rightarrow$ Work $= \int_C \mathbf{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$.

Example: $\mathbf{F} = \langle yz, xz, xy\rangle$. C: $x = t^3$, $y = t^2$, $z = t$. $0 \le t \le 1$. Then $dx = 3t^2dt$, $dy = 2tdt$, $dz = dt$ and substitute:

$$\int_C \mathbf{F}\cdot d\vec{r} = \int_C yzdx + xzdy + xydz = \int_0^1 6t^5 dt = 1$$

(In general, express (x, y, z) in terms of a *single* parameter: 1 degree of freedom)

Same $\mathbf{F}$, curve C' = segments from $(0,0,0)$ to $(1,0,0)$ to $(1,1,0)$ to $(1,1,1)$. In the xy-plane, $z = 0 \implies \mathbf{F} = xy\hat{\boldsymbol{k}}$, so $\mathbf{F}\cdot d\vec{r} = 0$, no work. For the last segment, $x = y = 1$, $dx = dy = 0$, so $\mathbf{F} = \langle z, z, 1\rangle$ and $d\vec{r} = \langle 0, 0, dz\rangle$. We get $\int_0^1 1\,dz = 1$.

Both give the same answer because $\mathbf{F}$ is conservative, in fact $\mathbf{F} = \nabla(xyz)$.

Recall the fundamental theorem of calculus for line integrals:

$$\int_{P_0}^{P_1} \nabla f\cdot d\vec{r} = f(P_1) - f(P_0).$$

Gradient fields.

$\mathbf{F} = \langle P, Q, R\rangle = \langle f_x, f_y, f_z\rangle$?

Then $f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$, $f_{yz} = f_{zy}$, so $P_y = Q_x$, $P_z = R_x$, $Q_z = R_y$.

Theorem: $\mathbf{F}$ is a gradient field if and only if these equalities hold (assuming defined in whole space or simply connected region)

Example: for which a, b is $axy\hat{\boldsymbol{\imath}} + (x^2 + z^3)\hat{\boldsymbol{\jmath}} + (byz^2 - 4z^3)\hat{\boldsymbol{k}}$ a gradient field?

$P_y = ax = 2x = Q_x$ so $a = 2$; $P_z = 0 = 0 = R_x$; $Q_z = 3z^2 = bz^2 = R_y$ so $b = 3$.

Systematic method to find a potential: (carried out on above example)

$f_x = 2xy$, $f_y = x^2 + z^3$, $f_z = 3yz^2 - 4z^3$:

$f_x = 2xy \Rightarrow f(x, y, z) = x^2y + g(y, z)$.

$f_y = x^2 + g_y = x^2 + z^3 \Rightarrow g_y = z^3 \Rightarrow g(y, z) = yz^3 + h(z)$, and $f = x^2y + yz^3 + h(z)$.

$f_z = 3yz^2 + h'(z) = 3yz^2 - 4z^3 \Rightarrow h'(z) = -4z^3 \Rightarrow h(z) = -z^4 + c$, and $f = x^2y + yz^3 - z^4 + c$.

Other method: $f(x_1, y_1, z_1) = f(0,0,0) + \int_{P_0}^{P_1} \mathbf{F}\cdot d\vec{r}$ (use a curve that gives an easy computation, e.g. 3 segments parallel to axes).

Curl: encodes by how much $\mathbf{F}$ fails to be conservative.

$\text{curl}\,\langle P, Q, R\rangle = (R_y - Q_z)\hat{\boldsymbol{\imath}} + (P_z - R_x)\hat{\boldsymbol{\jmath}} + (Q_x - P_y)\hat{\boldsymbol{k}}$.

How to remember the formula? Use the del operator $\nabla = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z\rangle$.

Recall from last week that $\nabla \cdot \mathbf{F} = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z\rangle \cdot \langle P, Q, R\rangle = P_x + Q_y + R_z = \operatorname{div} \mathbf{F}$.

Now we have: $\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \operatorname{curl} \mathbf{F}$.

Interpretation of curl for velocity fields: curl measures *angular velocity.*

Example: rotation around z-axis at constant angular velocity ω (trajectories are circles centered on z-axis): $\boldsymbol{v} = \langle -\omega y, \omega x, 0\rangle$.

Then $\nabla \times \boldsymbol{v} = \ldots = 0\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + (\omega + \omega)\hat{\boldsymbol{k}} = 2\omega\hat{\boldsymbol{k}}$. So length of curl = twice angular velocity, and direction = axis of rotation.

A general motion can be complicated, but decomposes into various effects.

• curl measures the *rotation* component of a complex motion.

18.02 Lecture 31. – **Thu, Nov 29, 2007**

Handouts: PS11 solutions, PS12.

Stokes' theorem is the 3D analogue of Green's theorem for work (in the same sense as the divergence theorem is the 3D analogue of Green for flux).

Recall $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$.

Stokes' theorem: if C is a *closed curve*, and S *any* surface bounded by C, then

$$\oint_C \mathbf{F} \cdot d\vec{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS.$$

Orientation: compatibility of an orientation of C with an orientation of S (changing orientation changes sign on both sides of Stokes).

Rule: if I walk along C in positive direction, with S to my left, then $\hat{\mathbf{n}}$ is pointing up. (Various examples shown.)

Another formulation (right-hand rule): if thumb points along C (1-D object), index finger towards S (2-D object), then middle finger points along $\hat{\mathbf{n}}$ (3-D object).

More examples shown.

Example: Stokes vs. Green. If S is a portion of xy-plane bounded by a curve C counterclockwise, then $\oint_C \mathbf{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy$, by Green this is equal to $\iint_S (Q_x - P_y)\,dx\,dy = \iint_S \nabla \times \mathbf{F} \cdot \hat{\boldsymbol{k}}\,dx\,dy = \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\,dS$, so Green and Stokes say the same thing in this example.

Remark. In Stokes' theorem we are free to choose any surface S bounded by C! (e.g. if C = circle, S could be a disk, a hemisphere, a cone, ...)

"Proof" of Stokes.

1) if C and S are in the xy-plane then the statement follows from Green.

2) if C and S are in an arbitrary plane: this also reduces to Green in the given plane. Green/Stokes works in any plane because of *geometric invariance* of work, curl and flux under rotations of space. They can be defined in purely geometric terms so as not to depend on the coordinate system (x, y, z); equivalently, we can choose coordinates (u, v, w) adapted to the given plane, and work

with those coordinates, the expressions of work, curl, flux will be the familiar ones replacing x, y, z with u, v, w.

3) in general, we can decompose S into small pieces, each piece is nearly flat (slanted plane); on each piece we have approximately work = flux by Green's theorem. When adding pieces, the line integrals over the inner boundaries cancel each other and we get the line integral over C; the flux integrals add up to flux through S.

Example: verify Stokes for $\mathbf{F} = z\hat{\imath} + x\hat{\jmath} + y\hat{\boldsymbol{k}}$, C = unit circle in xy-plane (counterclockwise), S = piece of paraboloid $z = 1 - x^2 - y^2$.

Direct calculation: $x = \cos\theta$, $y = \sin\theta$, $z = 0$, so $\oint_C \mathbf{F} \cdot d\vec{r} = \int_C z\,dx + x\,dy + y\,dz = \oint_C x\,dy = \int_0^{2\pi} \cos^2\theta\,d\theta = \pi$.

By Stokes: $\operatorname{curl}\mathbf{F} = \langle 1, 1, 1\rangle$, and $\hat{\mathbf{n}}\,dS = \langle -f_x, -f_y, 1\rangle dx\,dy = \langle 2x, 2y, 1\rangle\,dx\,dy$.

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\vec{S} = \iint \langle (2x + 2y + 1)\,dx\,dy = \iint 1\,dx\,dy = \text{area(disk)} = \pi.$$

($\iint x\,dx\,dy = 0$ by symmetry and similarly for y).

18.02 Lecture 32. – **Fri, Nov 30, 2007**

Stokes and path independence.

Definition: a region is simply connected if every closed loop C inside it bounds some surface S inside it.

Example: the complement of the z-axis is not simply connected (shown by considering a loop encircling the z-axis); the complement of the origin is simply connected.

Topology: uses these considerations to classify for example surfaces in space: e.g., the mathematical proof that a sphere and a torus are "different" surfaces is that the sphere is simply connected, the torus isn't (in fact it has two "independent" loops that don't bound).

Recall: if $\mathbf{F}$ is a gradient field then $\operatorname{curl}(\mathbf{F}) = 0$.

Conversely, Theorem: if $\nabla \times \mathbf{F} = 0$ in a *simply connected* region then $\mathbf{F}$ is conservative (so $\int \mathbf{F} \cdot d\vec{r}$ is path-independent and we can find a potential).

Proof: Assume R simply connected, $\nabla \times \mathbf{F} = 0$, and consider two curves C_1 and C_2 with same end points. Then $C = C_1 - C_2$ is a closed curve so bounds some S; $\int_{C_1} \mathbf{F} \cdot d\vec{r} - \int_{C_2} \mathbf{F} \cdot d\vec{r} = \oint_C \mathbf{F} \cdot d\vec{r} = \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\,dS = 0$.

Orientability. We can apply Stokes' theorem to any surface S bounded by C... provided that it has a well-defined normal vector! Counterexample shown: the Möbius strip. It's a one-sided surface, so we can't compute flux through it (no possible consistent choice of orientation of $\hat{\mathbf{n}}$). Instead, if we want to apply Stokes to the boundary curve C, we must find a two-sided surface with boundary C. (pictures shown).

Stokes and surface independence.

In Stokes we can choose any S bounded by C: so if a same C bounds two surfaces S_1, S_2, then $\oint_C \mathbf{F} \cdot d\vec{r} = \iint_{S_1} \operatorname{curl}\mathbf{F} \cdot \hat{\mathbf{n}}\,dS = \iint_{S_2} \operatorname{curl}\mathbf{F} \cdot \hat{\mathbf{n}}\,dS$? Can we prove directly that the two flux integrals are equal?

Answer: change orientation of S_2, then $S = S_1 - S_2$ is a closed surface with $\hat{\mathbf{n}}$ outwards; so we can apply the divergence theorem: $\iint_S \operatorname{curl}\mathbf{F} \cdot \hat{\mathbf{n}}\,dS = \iiint_D \operatorname{div}(\operatorname{curl}\mathbf{F})\,dV$. But $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$, always. (Checked by calculating in terms of components of $\mathbf{F}$; also, symbolically: $\nabla \cdot (\nabla \times \mathbf{F}) = 0$, much like $u \cdot (u \times v) = 0$ for genuine vectors).

Review for Exam 4.

We've seen three types of integrals, with different ways of evaluating:

1) $\iiint f\,dV$ in rect., cyl., spherical coordinates (I re-explained the general setup and the formulas for dV); applications: center of mass, moment of inertia, gravitational attraction.

2) surface integrals, flux. Setting up $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\,dS$, by knowing formulas for $\hat{\mathbf{n}}\,dS$.

We have seen: planes parallel to coordinate planes (e.g. yz-plane: $\hat{\mathbf{n}} = \pm\hat{\boldsymbol{\imath}}$, $dS = dy\,dz$); spheres and cylinders ($\hat{\mathbf{n}}$ = straight out/in from center or axis; $dS = a\,dz\,d\theta$ for cylinders, $a^2 \sin\phi\,d\phi d\theta$ for spheres); if we can express $z = f(x,y)$, $\hat{\mathbf{n}}\,dS = \pm\langle -f_x, -f_y, 1\rangle dx\,dy$ (recall $\langle \dots \rangle$ is not $\hat{\mathbf{n}}$ and $dx\,dy$ is not dS); if S has a given normal vector $\vec{N}$ (e.g. if S is given by $g(x,y,z) = 0$), $\hat{\mathbf{n}}\,dS = \pm\vec{N}/(\vec{N} \cdot \hat{\boldsymbol{k}})\,dx\,dy$.

3) line integrals $\int_C \mathbf{F} \cdot d\vec{r}$ ($=\int_C P\,dx + Q\,dy + R\,dz$), evaluate by parameterizing C and expressing in terms of a single variable.

While these various types of integrals are completely different in terms of interpretation and method of evaluation, we've seen some theorems that establish bridges between them:

a) ($\iint$ vs $\iiint$) divergence theorem: S closed surface, $\hat{\mathbf{n}}$ outwards, then $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\,dS = \iiint_D (\operatorname{div} \mathbf{F})\,dV$.

b) ($\int$ vs $\iint$) Stokes' theorem: C closed curve bounding S compatibly oriented, then $\int_C \mathbf{F} \cdot d\vec{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\,dS$.

Both sides of these theorems are integrals of the types discussed above, and are evaluated by the usual methods! (even if the integrand happens to be a div or a curl).

In fact, another conceptually similar bridge exists between no integral at all and line integral: the fundamental theorem of calculus, $f(P_1) - f(P_0) = \int_C \nabla f \cdot d\vec{r}$.

One more topic: given $\mathbf{F}$ with $\operatorname{curl} \mathbf{F} = 0$, finding a potential function.