**18.02 Lecture** 29. – **Tue, Nov 20, 2007**

Recall statement of divergence theorem: $\iint_S \mathbf{F}\cdot d\vec{S} = \iiint_D \operatorname{div}\mathbf{F}\,dV$.

**Del operator.** $\nabla = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z\rangle$ (symbolic notation!)

$\nabla f = \langle \partial f/\partial x, \partial f/\partial y, \partial f/\partial z\rangle =$ gradient.

$\nabla\cdot\mathbf{F} = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z\rangle \cdot \langle P, Q, R\rangle = P_x + Q_y + R_z =$ divergence.

**Physical interpretation.** $\operatorname{div}\mathbf{F}$ = source rate = flux generated per unit volume. Imagine an incompressible fluid flow (i.e. a given mass occupies a fixed volume) with velocity $\mathbf{F}$, then $\iiint_D \operatorname{div}\mathbf{F}\,dV = \iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS =$ flux through $S$ is the net amount leaving $D$ per unit time = total amount of sources (minus sinks) in $D$.

**Proof of divergence theorem.** To show $\iint_S \langle P, Q, R\rangle\cdot d\vec{S} = \iiint_D (P_x + Q_y + R_z)\,dV$, we can separate into sum over components and just show $\iint_S R\hat{\boldsymbol{k}}\cdot d\vec{S} = \iiint_D R_z\,dV$ & same for $P$ and $Q$.

If the region $D$ is vertically simple, i.e. top and bottom surfaces are graphs, $z_1(x,y) \le z \le z_2(x,y)$, with $(x,y)$ in some region $U$ of $xy$-plane: r.h.s. is

$$\iiint_D R_z\,dV = \iint_U \Big(\int_{z_1(x,y)}^{z_2(x,y)} R_z\,dz\Big)\,dx\,dy = \iint_U (R(x,y,z_2(x,y)) - R(x,y,z_1(x,y))\,dx\,dy.$$

Flux through top: $d\vec{S} = \langle -\partial z_2/\partial x, -\partial z_2/\partial y, 1\rangle dx\,dy$, so $\iint_{\text{top}} R\hat{\boldsymbol{k}}\cdot d\vec{S} = \iint R(x,y,z_2(x,y))\,dx\,dy$.

Bottom: $d\vec{S} = -\langle -\partial z_1/\partial x, -\partial z_1/\partial y, 1\rangle dx\,dy$, so $\iint_{\text{bottom}} R\hat{\boldsymbol{k}}\cdot d\vec{S} = \iint -R(x,y,z_1(x,y))\,dx\,dy$.

Sides: sides are vertical, $\hat{\mathbf{n}}$ is horizontal, $\mathbf{F}$ is vertical so flux = 0.

Given any region $D$, decompose it into vertically simple pieces (illustrated for a donut). Then $\iiint_D$ = sum of pieces (clear), and $\iint_S$ = sum of pieces since the internal boundaries cancel each other.

**Diffusion equation:** governs motion of smoke in (immobile) air (dye in solution, ...)

The equation is: $\dfrac{\partial u}{\partial t} = k\nabla^2 u = k\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right)$,
where $u(x,y,z,t)$ = concentration of smoke; we'll also introduce $\mathbf{F}$ = flow of the smoke. It's also the heat equation ($u$ = temperature).

Equation uses two inputs:

1) Physics: $\mathbf{F} = -k\nabla u$ (flow goes from highest to lowest concentration, faster if concentration changes more abruptly).

2) Flux and quantity of smoke are related: if $D$ bounded by closed surface $S$, then $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = -\frac{d}{dt}\iiint_D u\,dV$. (flux out of $D$ = - variation of total amount of smoke inside $D$)

By differentiation under integral sign, the r.h.s. is $-\iiint_D \frac{\partial}{\partial t} u\,dV$ (This can be explained in terms of integral as a sum of $u(x_i, y_i, z_i, t)\Delta V_i$ and derivative of sum is sum of derivatives) and by divergence theorem the l.h.s. is $\iiint_D \operatorname{div}\mathbf{F}\,dV$. Dividing by volume of $D$, we get

$$-\frac{1}{vol(D)}\iiint_D \frac{\partial u}{\partial t}\,dV = \frac{1}{vol(D)}\iiint_D \operatorname{div}\mathbf{F}\,dV.$$

Same average values over any region; taking limit as $D$ shrinks to a point, get $\partial u/\partial t = -\operatorname{div}\mathbf{F}$.

Combining, we get the diffusion equation: $\partial u/\partial t = -\operatorname{div}\mathbf{F} = +k\operatorname{div}(\nabla u) = k\nabla^2 u$.