18.02 Multivariable Calculus
Fall 2007

18.02 Lecture 24. – **Tue, Nov 6, 2007**

Simply connected regions. [slightly different from the actual notations used]

Recall Green's theorem: if C is a closed curve around R counterclockwise then line integrals can be expressed as double integrals:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \operatorname{curl}(\vec{F})\, dA, \qquad \oint_C \vec{F}\cdot \hat{\boldsymbol{n}}\, ds = \iint_R \operatorname{div}(\vec{F})\, dA,$$

where $\operatorname{curl}(M\hat{\boldsymbol{\imath}} + N\hat{\boldsymbol{\jmath}}) = N_x - M_y$, $\operatorname{div}(P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}}) = P_x + Q_y$.

For Green's theorem to hold, $\vec{F}$ must be defined on the *entire* region R enclosed by C.

Example: (same as in pset): $\vec{F} = \dfrac{-y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}}{x^2+y^2}$, $C =$ unit circle counterclockwise, then $\operatorname{curl}(\vec{F}) = \dfrac{\partial}{\partial x}(\dfrac{x}{x^2+y^2}) - \dfrac{\partial}{\partial y}(\dfrac{-y}{x^2+y^2}) = \cdots = 0$. So, if we look at both sides of Green's theorem:

$$\oint_C \vec{F}\cdot d\vec{r} = 2\pi \quad \text{(from pset)}, \qquad \iint_R \operatorname{curl}\vec{F}\, dA = \iint_R 0\, dA = 0\,?$$

The problem is that R includes 0, where $\vec{F}$ is not defined.

Definition: a region R in the plane is simply connected if, given any closed curve in R, its interior region is entirely contained in R.

Examples shown.

So: Green's theorem applies safely when the domain in which $\vec{F}$ is defined and differentiable is simply connected: then we automatically know that, if $\vec{F}$ is defined on C, then it's also defined in the region bounded by C.

In the above example, can't apply Green to the unit circle, because the domain of definition of $\vec{F}$ is not simply connected. Still, we can apply Green's theorem to an annulus (picture shown of a curve $C' =$ unit circle counterclockwise + segment along x-axis + small circle around origin clockwise + back to the unit circle allong the x-axis, enclosing an annulus R'). Then Green applies and says $\oint_{C'} \vec{F}\cdot d\vec{r} = \iint_{R'} 0\, dA = 0$; but line integral simplifies to $\oint_{C'} = \int_C - \int_{C_2}$, where $C =$ unit circle, $C_2 =$ small circle / origin; so line integral is actually the same on C and C_2 (or any other curve encircling the origin).

Review for Exam 3.

2 main objects: double integrals and line integrals. Must know how to set up and evaluate.

Double integrals: drawing picture of region, taking slices to set up the iterated integral.

Also in polar coordinates, with $dA = r\, dr\, d\theta$ (see e.g. Problem 2; not done)

Remember: mass, centroid, moment of inertia.

For evaluation, need to know: usual basic integrals (e.g. $\int \frac{dx}{x}$); integration by substitution (e.g. $\displaystyle\int_0^1 \frac{t\, dt}{\sqrt{1+t^2}} = \int_1^2 \frac{du}{2\sqrt{u}}$, setting $u = 1+t^2$). Don't need to know: complicated trigonometric integrals (e.g. $\int cos^4\theta\, d\theta$), integration by parts.

Change of variables: recall method:

1) Jacobian: $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ c_x & v_y \end{vmatrix}$. Its absolute value gives ratio between $du\, dv$ and $dx\, dy$.

2) express integrand in terms of u, v.

3) set up bounds in uv-coordinates by drawing picture. The actual example on the test will be reasonably simple (constant bounds, or circle in uv-coords).

Line integrals: $\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \hat{\boldsymbol{T}}\, ds = \int_C M\, dx + N\, dy$. To evaluate, express both x, y in terms of a single parameter and substitute.

Special case: gradient fields. Recall: $\vec{F}$ is conservative $\Leftrightarrow \int \vec{F} \cdot d\vec{r}$ is path independent $\Leftrightarrow \vec{F}$ is the gradient of some potential $f \Leftrightarrow \operatorname{curl} \vec{F} = 0$ (i.e. $N_x = M_y$).

If this is the case, then we can look for a potential using one of the two methods (antiderivatives, or line integral); and we can then use the FTC to avoid calculating the line integral. (cf. Problem 3).

Flux: $\int_C \vec{F} \cdot \hat{\boldsymbol{n}}\, ds$ $(= \int_C -Q\, dx + P\, dy)$. Geometric interpretation.

Green's theorem (in both forms) (already written at beginning of lecture).

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Handouts: Exam 3 solutions.

Triple integrals: $\displaystyle\iiint_R f\, dV$ (dV = volume element).

Example 1: region between paraboloids $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$ (picture drawn), e.g. volume of this region: $\displaystyle\iiint_R 1\, dV = \int_?^? \int_?^? \int_{x^2+y^2}^{4-x^2-y^2} dz\, dy\, dx.$

To set up bounds, (1) for fixed (x, y) find bounds for z: here lower limit is $z = x^2+y^2$, upper limit is $z = 4-x^2-y^2$; (2) find the shadow of R onto the xy-plane, i.e. set of values of (x, y) above which region lies. Here: R is widest at intersection of paraboloids, which is in plane $z = 2$; general method: for which (x, y) is z on top surface $>$ z on bottom surface? Answer: when $4 - x^2 - y^2 > x^2 - y^2$, i.e. $x^2 + y^2 < 2$. So we integrate over a disk of radius $\sqrt{2}$ in the xy-plane. By usual method to set up double integrals, we finally get:

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} dz\, dy\, dx.$$

Evaluation would be easier if we used polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$: then

$$V = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r^2}^{4-r^2} dz\, r\, dr\, d\theta.$$

(evaluation easy, not done).

Cylindrical coordinates. (r, θ, z), $x = r\cos\theta$, $y = r\sin\theta$. r measures distance from z-axis, θ measures angle from xz-plane (picture shown).

Cylinder of radius a centered on z-axis is $r = a$ (drawn); $\theta = 0$ is a vertical half-plane (not drawn).

Volume element: in rect. coords., $dV = dx\, dy\, dz$; in cylindrical coords., $dV = r\, dr\, d\theta\, dz$. In both cases this is justified by considering a small box with height Δz and base area ΔA, then volume is $\Delta V = \Delta A\, \Delta z$.

Applications: Mass: $M = \iiint_R \delta\, dV$.

Average value of f over R: $\bar{f} = \dfrac{1}{Vol}\iiint_R f\,dV$; weighted average: $\bar{f} = \dfrac{1}{Mass}\iiint_R f\,\delta\,dV$.

In particular, center of mass: $(\bar{x},\bar{y},\bar{z})$ where $\bar{x} = \dfrac{1}{Mass}\iiint_R x\,\delta\,dV$.

(Note: can sometimes avoid calculation using symmetry, e.g. in above example $\bar{x} = \bar{y} = 0$).

Moment of inertia around an axis: $I = \iiint_R (\text{distance from axis})^2\,\delta\,dV$.

About z-axis: $I_z = \iiint_R r^2\,\delta\,dV = \iiint_R (x^2+y^2)\,\delta\,dV$. (consistent with I_0 in 2D case)

Similarly, about x and y axes: $I_x = \iiint_R (y^2+z^2)\,\delta\,dV$, $I_y = \iiint_R (x^2+z^2)\,\delta\,dV$
(setting $z = 0$, this is consistent with previous definitions of I_x and I_y for plane regions).

Example 2: moment of inertia I_z of solid cone between $z = ar$ and $z = b$ ($\delta = 1$) (picture drawn):

$$I_z = \iiint_R r^2\,dV = \int_0^b \int_0^{2\pi} \int_0^{z/a} r^2\, r\,dr\,d\theta\,dz \quad \left(= \frac{\pi b^5}{10a^4}\right).$$

(I explained how to find bounds in order $dr\,d\theta\,dz$: first we fix z, then slice for given z is the disk bounded by $r = z/a$; the first slice is $z = 0$, the last one is $z = b$).

Example 3: volume of region where $z > 1 - y$ and $x^2 + y^2 + z^2 < 1$? Pictures drawn: in space, slice by yz-plane, and projection to xy-plane.

The bottom surface is the plane $z = 1 - y$, the upper one is the sphere $z = \sqrt{1-x^2-y^2}$. So inner is $\displaystyle\int_{1-y}^{\sqrt{1-x^2-y^2}} dz$. The shadow on the xy-plane = points where $1 - y < \sqrt{1-x^2-y^2}$, i.e. squaring both sides, $(1-y)^2 < 1 - x^2 - y^2$ i.e. $x^2 < 2y - 2y^2$, i.e. $-\sqrt{2y-2y^2} < x < \sqrt{2y-2y^2}$. So we get:

$$\int_0^1 \int_{-\sqrt{2y-2y^2}}^{\sqrt{2y-2y^2}} \int_{1-y}^{\sqrt{1-x^2-y^2}} dz\,dx\,dy.$$

Bounds for y: either by observing that $x^2 < 2y - y^2$ has solutions iff $2y - y^2 > 0$, i.e. $0 < y < 1$, or by looking at picture where clearly leftmost point is on z-axis ($y = 0$) and rightmost point is at $y = 1$.