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18.02 Multivariable Calculus
Fall 2007

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18.02 Lecture 1. – Thu, Sept 6, 2007

Handouts: syllabus; PS1; flashcards.

Goal of multivariable calculus: tools to handle problems with several parameters – functions of several variables.

Vectors. A vector (notation: \vec{A}) has a direction, and a length ($|\vec{A}|$). It is represented by a directed line segment. In a coordinate system it's expressed by components: in space, $\vec{A} = \langle a_1, a_2, a_3 \rangle = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$. (Recall in space x -axis points to the lower-left, y to the right, z up).

Scalar multiplication

Formula for length? Showed picture of $\langle 3, 2, 1 \rangle$ and used flashcards to ask for its length. Most students got the right answer ($\sqrt{14}$).

You can explain why $|\vec{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ by reducing to the Pythagorean theorem in the plane (Draw a picture, showing \vec{A} and its projection to the xy -plane, then derived $|\vec{A}|$ from length of projection + Pythagorean theorem).

Vector addition: $\vec{A} + \vec{B}$ by head-to-tail addition: Draw a picture in a parallelogram (showed how the diagonals are $\vec{A} + \vec{B}$ and $\vec{B} - \vec{A}$); addition works componentwise, and it is true that $\vec{A} = 3\hat{i} + 2\hat{j} + \hat{k}$ on the displayed example.

Dot product.

Definition: $\vec{A} \cdot \vec{B} = a_1b_1 + a_2b_2 + a_3b_3$ (a scalar, not a vector).

Theorem: geometrically, $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos \theta$.

Explained the theorem as follows: first, $\vec{A} \cdot \vec{A} = |\vec{A}|^2 \cos 0 = |\vec{A}|^2$ is consistent with the definition. Next, consider a triangle with sides \vec{A} , \vec{B} , $\vec{C} = \vec{A} - \vec{B}$. Then the law of cosines gives $|\vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}| \cos \theta$, while we get

$$|\vec{C}|^2 = \vec{C} \cdot \vec{C} = (\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}) = |\vec{A}|^2 + |\vec{B}|^2 - 2\vec{A} \cdot \vec{B}.$$

Hence the theorem is a vector formulation of the law of cosines.

Applications. 1) computing lengths and angles: $\cos \theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}$.

Example: triangle in space with vertices $P = (1, 0, 0)$, $Q = (0, 1, 0)$, $R = (0, 0, 2)$, find angle at P :

$$\cos \theta = \frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{|\overrightarrow{PQ}||\overrightarrow{PR}|} = \frac{\langle -1, 1, 0 \rangle \cdot \langle -1, 0, 2 \rangle}{\sqrt{2}\sqrt{5}} = \frac{1}{\sqrt{10}}, \quad \theta \approx 71.5^\circ.$$

Note the sign of dot product: positive if angle less than 90° , negative if angle more than 90° , zero if perpendicular.

2) detecting orthogonality.

Example: what is the set of points where $x + 2y + 3z = 0$? (possible answers: empty set, a point, a line, a plane, a sphere, none of the above, I don't know).

Answer: plane; can see "by hand", but more geometrically use dot product: call $\vec{A} = \langle 1, 2, 3 \rangle$, $P = (x, y, z)$, then $\vec{A} \cdot \overrightarrow{OP} = x + 2y + 3z = 0 \Leftrightarrow |\vec{A}||\overrightarrow{OP}| \cos \theta = 0 \Leftrightarrow \theta = \pi/2 \Leftrightarrow \vec{A} \perp \overrightarrow{OP}$. So we get the plane through O with normal vector \vec{A} .

18.02 Lecture 2. – Fri, Sept 7, 2007

We've seen two applications of dot product: finding lengths/angles, and detecting orthogonality. A third one: finding components of a vector. If $\hat{\mathbf{u}}$ is a unit vector, $\vec{A} \cdot \hat{\mathbf{u}} = |\vec{A}| \cos \theta$ is the component of \vec{A} along the direction of $\hat{\mathbf{u}}$. E.g., $\vec{A} \cdot \hat{\mathbf{i}} =$ component of \vec{A} along x -axis.

Example: pendulum making an angle with vertical, force = weight of pendulum \vec{F} pointing downwards: then the physically important quantities are the components of \vec{F} along tangential direction (causes pendulum's motion), and along normal direction (causes string tension).

Area. E.g. of a polygon in plane: break into triangles. Area of triangle = $\frac{1}{2}$ base \times height = $\frac{1}{2} |\vec{A}| |\vec{B}| \sin \theta$ (= 1/2 area of parallelogram). Could get $\sin \theta$ using dot product to compute $\cos \theta$ and $\sin^2 + \cos^2 = 1$, but it gives an ugly formula. Instead, reduce to complementary angle $\theta' = \pi/2 - \theta$ by considering $\vec{A}' = \vec{A}$ rotated 90° counterclockwise (drew a picture). Then area of parallelogram = $|\vec{A}| |\vec{B}| \sin \theta = |\vec{A}'| |\vec{B}| \cos \theta' = \vec{A}' \cdot \vec{B}$.

Q: if $\vec{A} = \langle a_1, a_2 \rangle$, then what is \vec{A}' ? (showed picture, used flashcards). Answer: $\vec{A}' = \langle -a_2, a_1 \rangle$. (explained on picture). So area of parallelogram is $\langle b_1, b_2 \rangle \cdot \langle -a_2, a_1 \rangle = a_1 b_2 - a_2 b_1$.

Determinant. Definition: $\det(\vec{A}, \vec{B}) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$.

Geometrically: $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \pm$ area of parallelogram.

The sign of 2D determinant has to do with whether \vec{B} is counterclockwise or clockwise from \vec{A} , without details.

Determinant in space: $\det(\vec{A}, \vec{B}, \vec{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$.

Geometrically: $\det(\vec{A}, \vec{B}, \vec{C}) = \pm$ volume of parallelepiped. Referred to the notes for more about determinants.

Cross-product. (only for 2 vectors in space); gives a vector, not a scalar (unlike dot-product).

Definition: $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{\mathbf{i}} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$.

(the 3x3 determinant is a *symbolic* notation, the actual formula is the expansion).

Geometrically: $|\vec{A} \times \vec{B}| =$ area of space parallelogram with sides \vec{A} , \vec{B} ; direction = normal to the plane containing \vec{A} and \vec{B} .

How to decide between the two perpendicular directions = right-hand rule. 1) extend right hand in direction of \vec{A} ; 2) curl fingers towards direction of \vec{B} ; 3) thumb points in same direction as $\vec{A} \times \vec{B}$.

Flashcard Question: $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = ?$ (answer: $\hat{\mathbf{k}}$, checked both by geometric description and by calculation).

Triple product: volume of parallelepiped = area(base) \cdot height = $|\vec{B} \times \vec{C}| (\vec{A} \cdot \hat{\mathbf{n}})$, where $\hat{\mathbf{n}} = \vec{B} \times \vec{C} / |\vec{B} \times \vec{C}|$. So volume = $\vec{A} \cdot (\vec{B} \times \vec{C}) = \det(\vec{A}, \vec{B}, \vec{C})$. The latter identity can also be checked directly using components.