18.02 Multivariable Calculus
Fall 2007

18.02 Practice Exam 4A - Solutions

1a)

$$M_y = e^x z = N_x$$
$$M_z = e^x y = P_x$$
$$N_z = e^x + 2y = P_y$$

1b) We begin with

$$\begin{cases} f_x = e^x yz \\ f_y = e^x z + 2yz \\ f_z = e^x y + y^2 + 1 \end{cases}$$

Integrating f_x we get $f = e^x yz + g(y,z)$. Differentiating and comparing with the above equations we get

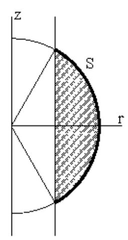
$$\begin{cases} f_y = e^x z + g_y \\ f_z = e^x y + g_z \end{cases} \rightarrow \begin{cases} g_y = 2yz \\ g_z = y^2 + 1 \end{cases}$$

Integrating g_y we get $g = y^2 z + h(z)$. Then $g_z = y^2 + h'(z)$ so comparing with the second equation above we get $h'(z) = 1$. Hence $h = z + C$. Putting everything together we get

$$f = e^x yz + y^2 z + z + C$$

1c) $N_z = 0$ and $P_y = 1$ hence the field is not conservative.

2a) Consider the figure



$\vec{n} = \frac{1}{2}(x, y, z)$ hence

$$\vec{F} \cdot \vec{n} = (y, -x, z) \cdot \frac{(x,y,z)}{2} = \frac{z^2}{2}$$

$z = 2\cos\phi$ and $dS = 2^2 \sin\phi \, d\phi \, d\theta$ hence we get

$$\int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{4\cos^2\phi}{2} 4\sin\phi \, d\phi \, d\theta = 16\pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \cos^2\phi \sin\phi d\phi = -16\pi \left[\frac{\cos^3\phi}{3}\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} = 4\sqrt{3}\pi$$

2b) $\vec{n} = \pm(x, y, 0)$ hence $\vec{F} \cdot \vec{n} = 0$. So the flux is 0.

2c) $div\vec{F} = 1$ hence

$$Vol(R) = \iiint_R 1 \, dV = \iiint_R div\vec{F} \, dV = \iint_S \vec{F} \cdot \vec{n} \, dS + \iint_{\text{Cylinder}} \vec{F} \cdot \vec{n} \, dS = 4\sqrt{3}\pi$$

3a) C is given by the equations $x^2 + y^2 + z^2 = 2$ and $z = 1$. So $x^2 + y^2 = 1$. Parametrization:

$$x = \cos t \qquad y = \sin t \qquad z = 1$$
$$dx = -\sin t \, dt \qquad dy = \cos t \, dt \qquad dz = 0$$

So

$$I = \int_0^{2\pi} (-\cos t \sin t + \sin t \cos t)dt = 0$$

3b)

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ xz & y & y \end{vmatrix} = \hat{\imath} + x\hat{\jmath}$$

3c) By Stokes theorem

$$\oint_C \vec{F} \cdot \vec{dr} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} dS$$

$\vec{n}$ is the normal pointing upward hence

$$\oint_C \vec{F} \cdot \vec{dr} = \iint_S (1, x, 0) \cdot \frac{(x, y, z)}{\sqrt{2}} dS = \iint_S \frac{x + xy}{\sqrt{2}} dS$$

4) $div\vec{F} = 0$ hence

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_R div\vec{F}\, dV = 0$$

5a)

$$z = (x^2 + y^2 + z^2)^2 \geq 0$$

5b) $z = \rho \cos \phi$ and $x^2 + y^2 + z^2 = \rho^2$ hence $\rho \cos \phi = \rho^4$. Canceling ρ we get $\cos \phi = \rho^3$.

5c)

$$\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{(\cos \phi)^{\frac{1}{3}}} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

6) The flux is upward so

$$\vec{n}\, dS = +(-f_x, -f_y, 1)\, dx\, dy = (-y, -x, 1)\, dx\, dy$$

$(f = xy)$. Hence

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_{x^2+y^2<1} (y, x, z) \cdot (-y, -x, 1)\, dx\, dy = \iint_{x^2+y^2<1} (-y^2 - x^2 + xy)\, dx\, dy$$

where we substituted $z = xy$. Using polar coordinates we get

$$\int_0^{2\pi} \int_0^1 (-r^2 + r^2 \cos \theta \sin \theta) r dr d\theta$$

- Inner: $\int_0^1 (-r^2 + r^2 \cos \theta \sin \theta) r dr = \frac{1}{4}(\cos \theta \sin \theta - 1)$
- Outer: $\int_0^{2\pi} \frac{1}{4}(\cos \theta \sin \theta - 1) d\theta = \frac{1}{4} \left[\frac{\sin^2 \theta}{2} - \theta \right]_0^{2\pi} = -\frac{\pi}{2}$