Confirming an Integral Converges

Use limit comparison to show that $\int_1^\infty \frac{dx}{(5x+2)^2}$ is finite.

Solution

Earlier we showed that $\int_1^\infty \frac{dx}{(5x+2)^2} = \frac{1}{35}$. While this calculation was not terribly difficult, many calculations are. Limit comparison provides us with a useful technique for dealing with unpleasant integrals.

We know that if $f(x) \sim g(x)$ as x goes to infinity then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ either both converge or both diverge.

Our first step in solving this function is to find a function $f(x)$ that is comparable to our given function $g(x) = \frac{1}{(5x+2)^2}$. We might algebraically expand the denominator and then discard its lower degree terms, letting $f(x) = \frac{1}{25x^2}$ or we might think back to our calculation of this integral and try $f(x) = \frac{1}{x^2}$; it turns out that the former strategy is the correct one.

To check that $f(x) = \frac{1}{25x^2} \sim \frac{1}{(5x+2)^2} = g(x)$ we must show that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{\frac{1}{25x^2}}{\frac{1}{(5x+2)^2}} \\
&= \lim_{x\to\infty} \frac{(5x+2)^2}{25x^2} \\
&= \lim_{x\to\infty} \frac{25x^2+20x+4}{25x^2} \\
&= \lim_{x\to\infty} \left(1 + \frac{4}{5x} + \frac{4}{25x^2}\right) \\
\lim_{x\to\infty} \frac{f(x)}{g(x)} &= 1
\end{aligned}$$

(If we had chosen to let $f(x) = \frac{1}{(5x+2)^2}$ and $g(x) = \frac{1}{25x^2}$, we would have wound up trying to simplify $\frac{25x^2}{(5x+2)^2}$. Because the reciprocal of this rational expression is much easier to work with, our best course of action at this point is to go back to the beginning and reverse the roles of $f(x)$ and $g(x)$.)

In the end we confirm that $\frac{1}{25x^2} \sim \frac{1}{(5x+2)^2}$ and so that $\int_1^\infty \frac{dx}{(5x+2)^2}$ converges if and only if $\int_1^\infty \frac{dx}{25x^2}$ does. We might know that $\int_1^\infty \frac{dx}{x^k}$ converges whenever $k > 1$, or we could evaluate the simpler integral:

$$\begin{aligned}\int_1^\infty \frac{dx}{25x^2} &= \frac{1}{25}\int_1^\infty x^{-2}\,dx \\ &= \frac{1}{25}\left[-x^{-1}\right]_1^\infty \\ &= 0 - \left(-\frac{1}{25}\right) = \frac{1}{25}\end{aligned}$$

We conclude that if $g(x) = \dfrac{1}{(5x+2)^2}$ and $f(x) = \dfrac{1}{25x^2}$, then $f(x) \sim g(x)$ and so:

$$\int_1^\infty \frac{dx}{(5x+2)^2}$$

must converge because:

$$\int_1^\infty \frac{dx}{25x^2} = \frac{1}{25}$$

converges.