Example: $\displaystyle\int_1^\infty \frac{dx}{x^p}$

We know that $\int_1^\infty \frac{dx}{x}$ diverges. Next we'll find $\int_1^\infty \frac{dx}{x^p}$ for any value of p; we'll see that $p = 1$ is a borderline when we do this calculation.

$$\begin{aligned}
\int_1^\infty \frac{dx}{x^p} &= \int_1^\infty x^{-p}dx \\
&= \left.\frac{x^{-p+1}}{-p+1}\right|_1^\infty \\
&= \frac{\infty^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1} \\
&= \frac{\infty^{-p+1}}{-p+1} + \frac{1}{p-1}
\end{aligned}$$

Remember that the ∞ in this expression is shorthand for "a number approaching infinity".

When we think about raising a very large number to the $p+1$ power we see that there are two cases that split exactly at $p = 1$. When $p = 1$, the exponent is zero and so is the denominator; the expression doesn't make any sense. For all other values of p the expression makes sense and the value of the integral depends on whether $-p + 1$ is positive or negative.

$$\frac{\infty^{-p+1}}{-p+1} \quad \text{is infinite when } -p+1 > 0$$

and

$$\frac{\infty^{-p+1}}{-p+1} \quad \text{is zero when } -p+1 < 0.$$

Check this yourself — this is the sort of problem that will be on the exam.

Conclusion: Combining this with our previous example we see that:

$$\int_1^\infty \frac{dx}{x^p} \quad \text{diverges if } p \leq 1$$

and

$$\int_1^\infty \frac{dx}{x^p} \quad \text{converges to } \tfrac{1}{p-1} \text{ if } p > 1.$$

Notice that when $p = 1$ our formula for the antiderivative is wrong; the antiderivative is $\ln x$ and not $\frac{x^{-p+1}}{-p+1}$. We really needed to do three separate calculations to compute the value of this integral: one for $p < 1$, one for $p = 1$ and one for $p > 1$.