

**Example:**  $\int_0^\infty e^{-kx} dx$  ( $k > 0$ )

This is the most fundamental, by far, of the improper integrals. We start by calculating  $\int_0^N e^{-kx} dx$ :

$$\begin{aligned}\int_0^N e^{-kx} dx &= -\frac{1}{k}e^{-kx} \Big|_0^N \\ &= -\frac{1}{k}e^{-kN} - \left(-\frac{1}{k}e^0\right) \\ &= -\frac{1}{k}e^{-kN} + \frac{1}{k}\end{aligned}$$

As  $N$  goes to infinity the  $\frac{1}{k}$  does not change, but  $-\frac{1}{k}e^{-kN}$  gets closer and closer to zero. (This is only true if  $k$  is positive!) So

$$\int_0^\infty e^{-kx} dx = \lim_{N \rightarrow \infty} \int_0^N e^{-kx} dx = \frac{1}{k}.$$

We can abbreviate this calculation as follows:

$$\begin{aligned}\int_0^\infty e^{-kx} dx &= -\frac{1}{k}e^{-kx} \Big|_0^\infty \\ &= -\frac{1}{k}e^{-\infty} - \left(-\frac{1}{k}e^0\right) \quad (\text{using the fact that } k > 0) \\ &= -0 + \frac{1}{k} \\ &= \frac{1}{k}\end{aligned}$$

**Question:** What if the limit is infinity?

**Answer:** Good question. There's a difference between the limit existing and the limit being infinite. Where our definition of improper integral says "if this limit exists" it means "exists and is finite"; if infinite limits are allowed they're mentioned explicitly, as in l'Hôpital's rule.

There is another part of this subject which we will not study here. If  $f$  changes sign (e.g.  $f(x) = \frac{\sin x}{x}$ ) there can be some cancellation in the integral as  $f$  oscillates. Sometimes the limit exists, but the total area enclosed above and below the  $x$ -axis is infinite. In order to avoid this possibility we require that  $f(x) > 0$ .

### Physical Interpretation

The number of radioactive particles in some radioactive substance that decay in time  $0 \leq t \leq T$  is given (on average) by:

$$\int_0^T Ae^{-kt} dt.$$

If we let  $T$  go to infinity we get:

$$\int_0^{\infty} Ae^{-kt} dt = \frac{A}{k} = \text{total number of particles.}$$

The notion that  $T$  goes to infinity is an idealization; we're not actually going to wait forever for the substance to decay. However, it's useful for us to write down and use this quantity even if it's not physically realistic. One reason to use this value as opposed to those for finite time intervals is that  $\frac{A}{k}$  is much easier to work with than  $-\frac{A}{k}e^{-kT} + \frac{A}{k}$ .

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18.01SC Single Variable Calculus  
Fall 2010

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