$\lim_{x \to 0} \dfrac{\sin x}{1 - \cos x}$

In this problem attempt to evaluate:

$$\lim_{x \to 0} \frac{\sin x}{1 - \cos x}$$

using approximation.

a) Substitute linear approximations for $\sin x$ and $\cos x$ into this expression. Can you tell what happens in the limit?

b) Substitute quadratic approximations for $\sin x$ and $\cos x$ into this expression. Can you tell what happens in the limit?

Solution

If we replace x by 0 in the quotient $\dfrac{\sin x}{1 - \cos x}$, the result is of the form $\frac{0}{0}$. We can use approximation to study this interesting limit; the results are similar to those we get from l'Hôpital's rule.

a) Substitute linear approximations for $\sin x$ and $\cos x$ into this expression. Can you tell what happens in the limit?

   Recall that the quadratic approximations of $\sin x$ and $\cos x$ near $x = 0$ are:

$$\begin{aligned} \cos x &\approx 1 \\ \sin x &\approx x. \end{aligned}$$

   Thus,

$$\begin{aligned} \lim_{x \to 0} \frac{\sin x}{1 - \cos x} &\approx \lim_{x \to 0} \frac{x}{1 - 1} \\ &= \lim_{x \to 0} \frac{x}{0}. \end{aligned}$$

   Replacing $\cos x$ by its linear approximation makes the denominator equal to 0. We begin to have some idea that this rational function is badly behaved, but it's hard to draw a conclusion from this calculation.

b) Substitute quadratic approximations for $\sin x$ and $\cos x$ into this expression. Can you tell what happens in the limit?

   The quadratic approximations of $\sin x$ and $\cos x$ near $x = 0$ are:

$$\begin{aligned} \cos x &\approx 1 - \frac{1}{2}x^2 \\ \sin x &\approx x. \end{aligned}$$

So:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x}{1-\cos x} &\approx \lim_{x\to 0}\frac{x}{1-(1-\frac{1}{2}x^2)} \\ &= \lim_{x\to 0}\frac{x}{\frac{1}{2}x^2} \\ &= \lim_{x\to 0}\frac{2}{x}.\end{aligned}$$

We can now see that as x approaches 0 the denominator of $\dfrac{\sin x}{1-\cos x}$ approaches 0 more rapidly than the numerator, and so the value of the rational function "blows up" near $x = 0$.