Elementary Example of L'Hôpital's Rule

We begin by applying L'Hôpital's rule to a problem we could have solved earlier:

$$\lim_{x\to 1} \frac{x^{10}-1}{x^2-1}.$$

We listed some categories of limits at the beginning of the course; this falls into the category of "interesting limits" because if we just plug in $x = 1$ we get $\frac{0}{0}$. This is called an *indeterminate form*.

To find the limit using techniques we already know, we'd do the following:

$$\lim_{x\to 1} \frac{x^{10}-1}{x^2-1} = \lim_{x\to 1} \frac{(x^{10}-1)/(x-1)}{(x^2-1)/(x-1)}.$$

We could calculate $(x^{10} - 1)/(x - 1)$ using long division, but that's a long calculation. We can find this limit more quickly using calculus.

We've used calculus to understand a fraction in indeterminate form when we studied the difference quotient. If $f(x) = x^{10} - 1$, then $f(1) = 0$ and the difference quotient is:

$$\frac{f(x)-f(1)}{(x-1)} = \frac{x^{10}-1}{x-1}.$$

We know from our studies of difference quotients that:

$$\lim_{x\to 1} \frac{f(x)-f(1)}{(x-1)} = f'(1).$$

We conclude that:

$$\lim_{x\to 1} \frac{x^{10}-1}{x-1} = f'(1) = 10.$$

Our expression:

$$\frac{x^{10}-1}{x^2-1} = \frac{(x^{10}-1)/(x-1)}{(x^2-1)/(x-1)}$$

describes a ratio of difference quotients, so if $g(x) = x^2 - 1$ this line of reasoning tells us that:

$$\begin{aligned}
\lim_{x\to 1} \frac{x^{10}-1}{x^2-1} &= \lim_{x\to 1} \frac{(x^{10}-1)/(x-1)}{(x^2-1)/(x-1)} \\
&= \frac{\lim_{x\to 1}((x^{10}-1)/(x-1))}{\lim_{x\to 1}((x^2-1)/(x-1))} \\
&= \frac{f'(1)}{g'(1)} \\
&= \frac{10}{2} \\
&= 5.
\end{aligned}$$

Dividing by $x - 1$ and interpreting the fraction as a ratio of difference quotients enabled us to solve the problem by taking two easy derivatives and saved us from a lengthy exercise in long division.