Complete the Square

Compute the integral $\int \frac{dx}{\sqrt{2x - x^2}}$ by completing the square.

Solution

We wish to transform the expression $2x - x^2$ inside the square root into a form that is on our list of common trig substitutions. The subtraction of x^2 here suggests that we're aiming for an expression comparable to $\sqrt{b^2 - x^2}$.

We can now look up a formula for completing the square, recognize that $2x - x^2$ is part of $-1 + 2x - x^2 = -(x-1)^2$, or solve the following equation for a and c:

$$\begin{aligned} 2x - x^2 &= c - (x+a)^2 \\ -x^2 + 2x &= c - (x^2 + 2ax + a^2) \\ -x^2 + 2x &= -x^2 - 2ax + c - a^2 \end{aligned}$$

Equating like terms we get:

$$\begin{aligned} 2x = -2ax &\implies a = -1, \\ 0 = c - a^2 &\implies c = 1. \end{aligned}$$

In other words, $2x - x^2 = 1 - (x-1)^2$.

We can now set up our integral, which will contain the expression $\sqrt{1-(x-1)^2}$. Our summary of trig substitutions suggests substituting $u = b\sin\theta$ if the expression $\sqrt{b^2 - u^2}$ appears in an integral, so we will substitute:

$$x - 1 = \sin\theta, \qquad dx = \cos\theta\, d\theta.$$

$$\begin{aligned} \int \frac{dx}{\sqrt{2x - x^2}} &= \int \frac{dx}{\sqrt{1-(x-1)^2}} \\ &= \int \frac{\cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}} \\ &= \int \frac{\cos\theta\, d\theta}{\cos\theta} \\ &= \int d\theta \\ &= \theta + c \\ &= \arcsin(x-1) + c \end{aligned}$$

If we know $\frac{d}{dx}\arcsin v = \frac{1}{\sqrt{1-v^2}}$, this answer is easy to check.