

## Completing the Square

The last thing we need to discuss on the topic of trig substitution is completing the square. This is necessary when we want to integrate the square root of a quadratic whose form is not as simple as  $ax^2 + c$ .

**Example:**  $\int \frac{dx}{\sqrt{x^2 + 4x}}$

The integrand contains the square root of a quadratic, but it doesn't match any of the forms in our table of trig substitutions. What we need to do is use substitution to rewrite it in a form that *is* in our table.

We'll start by trying to write the quadratic in the form:

$$(x + a)^2 + c.$$

$$\begin{aligned}x^2 + 4x &= (x + a)^2 + c \\ &= x^2 + 2xa + a^2 + c\end{aligned}$$

Since the coefficient on  $x$  on the left must equal that on the right, it must be true that  $2xa = 4x$ ; i.e.  $a = 2$ . Similarly, we get  $a^2 + c = 0$  or  $c = -4$ . So:

$$x^2 + 4 = (x + 2)^2 - 4.$$

This process of eliminating the "middle term" using the square of a linear function is called *completing the square*. When we plug this into our integral, it becomes:

$$\int \frac{dx}{\sqrt{x^2 + 4x}} = \int \frac{dx}{\sqrt{(x + 2)^2 - 4}}$$

This is closer to the forms listed in our trig substitution table, but a direct substitution of  $u = x + 2$  brings us even closer:

$$\int \frac{dx}{\sqrt{x^2 + 4x}} = \int \frac{du}{\sqrt{u^2 - 4}}$$

We now consult our table and perform the trig substitution  $u = 2 \sec \theta$ ,  $du = 2 \sec \theta \tan \theta d\theta$ .

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2 + 4x}} &= \int \frac{2 \sec \theta \tan \theta d\theta}{\sqrt{4 \sec^2 \theta - 4}} \\ &= \int \frac{2 \sec \theta \tan \theta d\theta}{2\sqrt{\tan^2 \theta}} \\ &= \int \frac{\sec \theta \tan \theta d\theta}{\tan \theta} \\ &= \int \sec \theta d\theta \\ &= \ln(\sec \theta + \tan \theta) + c\end{aligned}$$

We've completed the integration, but we still need to reverse our two substitutions. We readily see that  $\sec \theta = \frac{u}{2}$ , and in the process of computing the integral we calculated that  $\sqrt{u^2 - 4} = 2 \tan \theta$ , so  $\tan \theta = \frac{\sqrt{u^2 - 4}}{2}$ . Hence we can avoid drawing a triangle and say:

$$\int \frac{dx}{\sqrt{x^2 + 4x}} = \ln \left( \frac{u}{2} + \frac{\sqrt{u^2 - 4}}{2} \right) + c$$

Finally, we replace  $u$  by  $x + 2$  to reverse the first substitution:

$$\begin{aligned} \int \frac{dx}{\sqrt{x^2 + 4x}} &= \ln \left( \frac{x + 2}{2} + \frac{\sqrt{(x + 2)^2 - 4}}{2} \right) + c \\ &= \ln \left( \frac{x + 2}{2} + \frac{\sqrt{x^2 + 4x}}{2} \right) + c \end{aligned}$$

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