Substitution Practice

Use a trigonometric substitution to integrate the function $f(x) = x\sqrt{x^2-9}$. Check your work by integration using the substitution $u = x^2$.

Solution

Referring to our trig substitution summary, we see that the recommended way to integrate an expression including $\sqrt{x^2-3^2}$ is to substitute $x = 3\sec\theta$, in which case $dx = 3\sec\theta\tan\theta\,d\theta$ and:

$$\begin{aligned}
\sqrt{x^2-9} &= \sqrt{(3\sec\theta)^2-9} \\
&= \sqrt{9\sec^2\theta-9} \\
&= 3\sqrt{\sec^2\theta-1} \\
\sqrt{x^2-9} &= 3\tan\theta.
\end{aligned}$$

We start by performing this substitution and simplifying:

$$\begin{aligned}
\int x\sqrt{x^2-9}\,dx &= \int (3\sec\theta)(3\tan\theta)3\sec\theta\tan\theta\,d\theta \\
&= 27\int \sec^2\theta\tan^2\theta\,d\theta.
\end{aligned}$$

At this point we substitute $u = \tan\theta$, so $du = \sec^2\theta\,d\theta$ and:

$$\begin{aligned}
\int x\sqrt{x^2-9}\,dx &= 27\int \sec^2\theta\tan^2\theta\,d\theta \\
&= 27\int u^2\,du \\
&= 27\frac{u^3}{3}+c \\
&= 9\tan^3\theta+c
\end{aligned}$$

We have an answer in terms of $\tan\theta$ and we want an answer in terms of x. We know that $x = 3\sec\theta$, or equivalently that $\sec\theta = \frac{x}{3}$. Keeping that fact in mind (or the fact that $\cos\theta = \frac{3}{x}$) we draw a right triangle with one angle equal to θ in which the side adjacent to θ has length 3 and the hypotenuse has length x. By the Pythagorean theorem, the side opposite the angle θ has length $\sqrt{x^2-9}$ and:

$$\tan\theta = \frac{\sqrt{x^2-9}}{3}.$$

We can now complete our calculation:

$$\int x\sqrt{x^2-9}\,dx = 9\tan^3\theta+c$$

$$
\begin{aligned}
&= 9\left(\frac{\sqrt{x^2-9}}{3}\right)^3 + c \\
\int x\sqrt{x^2-9}\,dx &= \frac{1}{3}(x^2-9)^{3/2} + c
\end{aligned}
$$

While the calculation above is correct, it is faster and more reliable to compute this integral via the substitution $u = x^2 - 9$. If we do this we have $du = 2x\,dx$ or $x\,dx = \frac{1}{2}\,du$ and:

$$
\begin{aligned}
\int x\sqrt{x^2-9}\,dx &= \int \sqrt{u}\,\frac{1}{2}\,du \\
&= \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + c \\
&= \frac{1}{3}u^{3/2} + c \\
\int x\sqrt{x^2-9}\,dx &= \frac{1}{3}(x^2-9)^{3/2} + c.
\end{aligned}
$$